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AUTHOR Haag, V. H.; And Others
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ABSTRACT

This is volume two of a two-volume manual for teachers using SMSG text materials for students in grades 7 and 8 whose mathematical talents are underdeveloped. The overall purpose for each of the chapters is described and the mathematical development detailed. Background information for key concepts, answers for all exercises in each chapter, and suggested test items are provided. Chapter topics include: linear measurement, area and volume, angles and parallels, polygons and prisms, circles, statistics and graphs, negative rational numbers, equations and inequalities, coordinates in the plane, real numbers, and scientific notation, decimals, and the metric system. (MN)

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INTRODUCTION TO SECONDARY SCHOOL MATHEMATICS

VOLUME 2

SCHOOL MATHEMATICS STUDY GROUP

YALE UNIVERSITY PRESS

Introduction to Secondary School Mathematics, Volume 2

Teacher's Commentary

REVISED EDITION

Prepared under the supervision of
a Panel consisting of:

V. H. Haag	Franklin and Marshall College
Mildred Keiffer	Cincinnati Board of Education
Oscar Schaaf	South Eugene High School, Eugene, Oregon
M. A. Sobel	Montclair State College, Upper Montclair, New Jersey
Marie Wilcox	Thomas Carr Howe High School, Indianapolis, Indiana
A. B. Willcox	Amherst College

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FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

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PREFACE TO TEACHERS

This text has been written for the students in Grade 8 whose mathematical talent is underdeveloped. It is intended for use only with those groups who have completed the material in Volume I of "Introduction to Secondary School Mathematics." The subject matter which is presented in these two volumes, Chapters 1-16, is essentially that which appears in the School Mathematics Study Group text: "Mathematics for Junior High School," Volume I. The material in Chapters 17-21 represents the subject matter considered of greatest significance in the School Mathematics Study Group text: "Mathematics for Junior High School," Volume II. Together, this is part of the body of mathematics which members of the School Mathematics Study Group believe is important for all educated citizens. It is also the mathematics which is important for the pre-college student as he prepares for advanced work in the field of mathematics and related subjects.

Within the group for which this material is intended there may be a large number of college capable students whose mathematical talent as yet has not been discovered. There may be others who heretofore have been insufficiently challenged. This text is not offered as appropriate content for the very slow non-college-bound student.

A number of guides were followed in the preparation of these materials, among which are the following:

- to adjust the reading level downward;
- to shorten the chapters and to provide variation from chapter to chapter in terms of content;
- to shorten sections within each chapter;
- to introduce new concepts through the use of concrete examples;
- to provide numerous illustrative examples;
- to include simple drill material in many of the problem sets;
- to provide chapter summaries, chapter reviews, and cumulative sets of problems.

The mathematics which appears in this text is not of the type normally called "business" or "vocational" mathematics, nor is it intended to serve as the content for a terminal course. Rather, as the title clearly states, this book continues the introduction to secondary school mathematics which was begun in grade seven. The text is intended to provide the student with many of the basic concepts necessary for further study.

It is the hope of the panel that this material will serve to awaken the interest of a large group of students who have ability in mathematics which has not yet been recognized. It is hoped also that an understanding of fundamental concepts can be built for those whose progress in mathematics has been blocked or hampered through rote learning or through an inappropriate curriculum.

The teacher should keep in mind that this text is still being used to test the hypothesis that material of this type can be taught to young people of the ability level previously described. Consequently the development should follow the text closely in terms of content as well as methodology in order that a fair evaluation of the material may be made.

The material in this text is presented in a manner different from the usual text at this level, and as previously indicated, it is written for a particular type of student. For these reasons some general suggestions for its use are offered below.

Reading. This text is written with the expectation that it can and will be read by the student. Many students are not accustomed to reading a mathematics book so that it will be necessary to assist them in learning how to make the most effective use of the book.

Experience has shown that it is most productive for the teacher to read with the class during the early part of the course. The teacher may read aloud while the students read silently. Later the teacher may start the reading with the class and then encourage them to continue the reading alone.

This technique is not recommended as a method of teaching reading as such, but rather as a method of helping the student discover that he can read a mathematics book.

There will be times when the student will need to reread the same passages several times. The teacher should suggest this, and see that time is made available for it.

The students of the ability level for which this text was written may not be able to read long passages with understanding. Some students may be able to read only a sentence or two at a time in the beginning. Consequently, it is important that the teacher stop often for class discussion.

It may prove helpful if students are asked to state, in their own words, the ideas which they have read; but the teacher should remember that some pupils may understand even though they cannot verbalize.

It must be observed that the teacher's objective is to convey to the students the ideas contained in the material. The teacher cannot permit reading retardation to inhibit or to undermine student interest in the content. The mathematics teacher cannot overcome serious pupil retardation in reading, but he can contribute to reading skill by pointing out to the student the need for rereading and giving careful consideration to the material. The use of a pencil and paper to draw diagrams and illustrate ideas should be encouraged.

Precision of Language. Ideally, pupils should be encouraged to express themselves accurately. Some pupils, however, have limited vocabulary resources. It is wise to encourage them to express themselves in their own words, meager as their contributions may be. The pupil's inadequate expression may then be refined by the teacher so that it is mathematically precise. The teacher must also recognize that extreme insistence upon precise formulation may interfere with thought patterns and act as a barrier to free expression.

Discussion Questions. Periodically the text provides discussion questions which are useful in helping to strengthen or emphasize basic concepts and understandings. These are especially useful in developing ideas in sections of the text where straight reading may be difficult. Therefore, where class discussion exercises are provided, they should be treated orally within the class period and not omitted.

Discovery Approach. A student usually gains a better understanding of a concept if he "discovers" the concept himself. The teacher must set the stage for the discovery approach. No textbook can do this because the text must give the student correct information to which he can refer and by which he can check his own ideas. Therefore, the approach will not be effective unless the pupil is encouraged to work through the development before he reads in the text the idea he was to "discover".

Students with limited ability should be given the opportunity to "discover" very simple ideas. Chapters 13 and 14 provide many settings in which this may be done by experimentation. Thus, in Chapter 13, the student discovers that vertical angles have the same measure. Again, in Chapter 14, he discovers that the sum of the measures of the angles of a triangle is 180. Both of these discoveries, as well as the others presented in these chapters, come as the outgrowth of laboratory experiments by means of which the student is guided to take the necessary measurements and arrive at a conclusion. The teacher should allow adequate class time for such experimentation in order to emphasize this aspect of mathematical discovery.

It is important for these students to have many experiences with an idea in order to develop meaning. In all cases the teacher will need to clarify the idea which the student has discovered and assist him in finding "his" idea in the text in correct mathematical language.

Exercises. The text has an ample supply of exercises. They are graded in most sections so that the most difficult are at the end. Some of the exercise lists, however, are developmental in nature and need to be treated sequentially. The teacher should be very cautious about making any omissions in such lists. In other lists the teacher may find it desirable to omit selected items.

Assignments. Assignments for this group should be quite definite and should normally concern only material which has been discussed in class so that the student may enjoy some measure of success in the preparation of it. Exercises which demand deeper vision, a higher degree of abstraction, or a preview into new material should be called "extra credit", or given some such notation, so that the student with below-average ability may omit this part of the assignment without a feeling of failure or frustration.

Testing. Students of the ability for which this text is written need to have short tests at frequent intervals--possibly one a week. These tests, like the assignments, should be flexible. The major portion of the test should cover material actually discussed in class with a few exercises for the more capable students included at the end. If the slower-learning students are not given some test questions which they can answer correctly, they may lose interest in the course and the opportunity to improve their mathematical background will be lost. They must be permitted to enjoy some measure of success.

Since the intent of this book is to emphasize grasp of ideas rather than memorization, the testing program should be geared accordingly. The teacher should be generous in accepting expression of ideas in the student's own words.

Extent of Course. The number of chapters studied will depend upon the class situation, the length of the class period, and the length of the school year.

Content. The title of the book indicates that the content provides an introduction to secondary school mathematics. Throughout the course, Volumes I and II, emphasis is placed upon mathematics as a method of reasoning. The structure of our decimal numeration system is examined and then the counting numbers, whole numbers, rational numbers, and negative numbers are successively introduced over this two year period.

The basic properties (field axioms) are intuitively developed as the successive sets of numbers are studied. The familiar computational procedures are shown to be valid because of the properties of the number system and of the operations employed.

The number line and the idea of presenting numbers as points on a line provide the basis for all graphing and for analytical geometry. The number line provides the motivation for order relations between numbers and for the invention of real numbers.

Measurement in terms of arbitrary units of length, area, volume and angle is examined so that the student can understand the various standard units of measure in common use today. The metric and English units and the relations between them are discussed.

The main purpose of geometry included in this text is to present intuitively the concepts of point, line, and plane and to reach agreement by inductive reasoning that certain statements concerning these concepts appear to be true. Some of these statements will appear in the formal geometry course as axioms. Others will be proved as theorems. A second purpose of the geometry in this book is to present an introduction to the process of deductive reasoning in geometry.

Summary. We hope that by introducing the pupil to simple number theory, the development of the real number system, aspects of non-metric and metric geometry, and the notions of ratio and proportion, in a carefully paced fashion which makes full use of a developmental approach, we shall be successful in attracting and retaining increased numbers of pupils for continued study of mathematics. We hope that appropriate mathematics, suitably taught, will awaken interest in pupils whose progress in traditional courses has seemed hopeless. The discovery and nurture of heretofore unidentified capacity for learning mathematics is one of the main purposes of this book.

Chapter 11

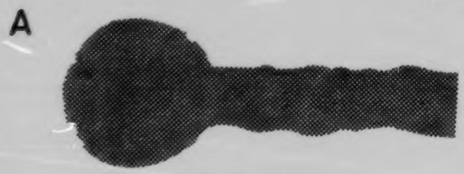
LINEAR MEASUREMENT

11-1. The Ruler.

A basic underlying idea in this chapter is that when some ideal geometrical segment has been chosen as a unit of length, then every ideal geometrical segment has an exact length. But these exact lengths are useful only in theoretical discussions. In the measurements of physical objects or the representation of segments drawn on a sheet of paper, many unavoidable sources of error occur. Some of these sources of error are listed below.

I. The endpoints of the segments we draw are not exactly specified. For example in the case of the segment $\overline{AB}$ shown below

if we magnify the segment in order to measure with greater accuracy, the picture may look like this:



It is clearly impossible to tell which point is denoted by A. The situation is the same even if A is represented by a very finely drawn dot. All we need in order to produce a picture like the last one is greater magnification.

II. The same remarks as in I apply to the ruler we are using as a measuring instrument.

III. Even if the points were ideally designated, our eye could not determine whether a point drawn on the paper and a point marked on the ruler coincide exactly.

IV. When calculating with lengths measured in decimal notation, we round off our lengths to a number of decimal places consistent with the accuracy of our measurements. That is, if a segment is measured to have length 5.0 units to the nearest tenth of a unit, then a segment one-third this long is said to have length 1.7 units instead of 1.66666666...... units.

V. Although our ruler is in theory a number line with infinitely many points all having distinct coordinates, in practice only finitely many points are marked on the ruler. Our measurements are then made by locating the marked point on the ruler closest to the endpoint of the segment.

In this chapter, only the last of these sources of error is discussed. The idea is met first in Section 11-1 on the ruler and expanded in Section 11-3 on precision and error. Our attitude, then, is that the segments we draw are ideal segments with exact lengths which, however, we can measure only to a precision determined by the spacing between the closest marks on our (ideal) ruler. The reason for this attitude is, of course, to avoid the confusion resulting from the discussion of several different types of error. If the subject does come up in class, the teacher may explain that the segments drawn in the book are only "representations" of exact segments.

The teacher is warned that the word error as used in this chapter does not denote a mistake. The measurements of the world's champion measurer would have the same "precision" and "greatest possible error" as those made by the student since these terms refer only to the scale on the ruler and not to the correctness with which the scale is read.

In addition to the errors mentioned above, two types of mistakes can occur.

I. Deciding that the endpoint of a segment is closer to one mark on the ruler when it is actually closer to another.

II. Incorrectly naming the coordinate of the mark on the ruler.

It will be noted that no provision has been made for handling the situation in which a point appears to be exactly half-way between two marks on the ruler. If the student can find no basis for preference for one of the two marks, he will have to shrug his shoulders and pick one of the two at random.

Answers to Exercises 11-1

Note: The students may refer to these "units" as "bars."

2. a. $3\frac{3}{4}$ units
   b. $4\frac{1}{4}$ units
   c. $3\frac{4}{4}$ units
   d. $3\frac{3}{4}$ units
   e. $4\frac{1}{2}$ units

3. $EF \approx 1$ unit
   $FG \approx 1\frac{1}{4}$ units
   $EG \approx 1\frac{4}{4}$ units

4. $\approx 4\frac{1}{4}$ units

6. a. $\approx 3\frac{5}{8}$ units
   b. $\approx 4\frac{1}{8}$ units
   c. $\approx 3\frac{7}{8}$ units
   d. $\approx 3\frac{6}{8}$ units
   e. $\approx 4\frac{3}{8}$ units

7. No. If the lengths of a and d are compared by transferring endpoints to the edge of a paper, then it is clear that their lengths are different, even though they have the same measurement to $\frac{1}{8}$ of the unit.
8. Answers will vary.
9. Answers will vary.

11-2. Standard Units of Length.

The pupils have had many exercises to demonstrate the meaning of measurement and the arbitrary nature of the unit. This section should establish the social need for units that are the same for the entire group. Mass production and the convenience of interchangeable parts provide a wealth of material for pointing out this need. The historical material should be of interest and show society's increasing need for standard units.

The metric system was treated briefly in this chapter because a study of the system as a whole was considered to be more appropriate for eighth grade. It should be stressed that our linear units are defined, by law, in terms of the metric units. The metric system is legal in the U.S. If class time permits, a discussion of the new definition of a meter as 1,650,763.73 times the wave length of orange light from Krypton 86 might be interesting. This standard for the meter is difficult to visualize, but has the advantage that it can be reproduced in any good scientific laboratory and is more precise than the platinum bar in France which was the former standard for the meter.

The approximate nature of measurement has been pointed out throughout the chapter. Continuing emphasis on the fact that measurements are not exact should be made, although this topic will be treated more formally in Section 3.

Comparing measurements of the same object made by different students and also measures determined by instruments marked with varying degrees of precision help develop the concept of the approximate nature of measurement.

Some 8th grade students have difficulty reading a ruler. Through making cardboard rulers and studying the increasingly fine divisions in Figure 11-2b, it is hoped that the poor students will overcome their difficulties while the better student is seeing the relationship with base 2. Some of the actual measuring of segments in Exercises 11-2c should be done in class so that the teacher can identify and work with students who have not mastered the use of the ruler.

In the hodge podge that is the English system of measures, there is a variety of standard units. Conversions from one unit to another cause a great deal of trouble.

Conversion is treated by means of ratio or proportion. This should be carefully developed in class and class practice should be provided to make sure that students understand what they are doing. Emphasis should be placed on the form

$$\frac{\text{number of feet in } \overline{AB}}{\text{number of centimeters in } \overline{AB}} = \frac{\text{number of feet in } \overline{CD}}{\text{number of centimeters in } \overline{CD}}.$$

Especial care should be made in avoiding interchanging numerators and denominators

$$\frac{\text{feet}}{\text{centimeters}} = \frac{\text{feet}}{\text{centimeters}}$$

$$\frac{\overline{AB}}{\overline{AB}} = \frac{\overline{CD}}{\overline{CD}}.$$

The fraction on the left involves only one segment, the fraction on the right involves the other segment. The numerators of both fractions involve the same units. The denominators of both fractions involve the same units. Common sense is an important aid in detecting gross errors in conversion.

Answers to Exercises 11-2a

1. The need to measure segments smaller than an inch.
2. Into 2 parts
3. Between the 1-inch and 5-inch marks each inch is divided into two parts. Then between the 2-inch and 5-inch marks each part is again divided into two parts, etc.
4. Each section is divided into two parts.
5. 4; 8; 16. (The third inch lies between 2 and 3.)
6. Theoretically, there is no limit to the number of divisions.

Discussion of Figure 11-2d should stress the advantage of comparing the end of a segment with a ruled mark rather than the end of a ruler. Especially is this an advantage if a ruler is improperly cut or has become worn. Practice in this means of measuring may need to be supplied. In Fig. 11-2c the length of $\overline{AB} \approx 2\frac{1}{2}$ inches.

Answers to Exercises 11-2b

1 and 2. Time and material will be saved if students are supplied with cut strips of cardboard. Four scales can be placed on two strips. Cardboard rulers can be checked with commercial rulers.

3. a. $\approx 2\frac{1}{2}$ in.;
   b. $\approx 3\frac{1}{2}$ in.;
   c. ≈ 2 in.;
   d. ≈ 4 in.;
   e. ≈ 3 in;

4. a. $\approx 2\frac{1}{2}$ in.;
   b. $\approx 3\frac{1}{2}$ (or $3\frac{1}{4}$) in.;
   c. $\approx 2\frac{1}{4}$ in.;
   d. $\approx 4\frac{1}{4}$ (or 4) in;
   e. ≈ 3 in.

5. a. $\approx 2\frac{1}{2}$ in.;
   b. $\approx 3\frac{3}{8}$ in.;
   c. $\approx 2\frac{1}{4}$ (or $2\frac{1}{8}$);
   d. $\approx 4\frac{1}{8}$ in.;
   e. ≈ 3 in (or $2\frac{7}{8}$ in.)

page 11-13: 11-2

6. Answers will vary.

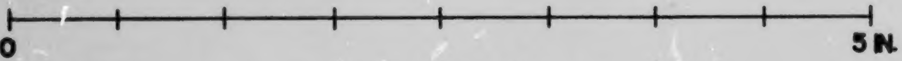
7. 8ths of an inch since the divisions come closer to matching the line segments.

Answers to Exercises 11-2c

1. (A) $\frac{3}{4}$" (C) $2\frac{7}{8}$" (E) $4\frac{3}{8}$"
   (B) $1\frac{5}{8}$" (D) $3\frac{3}{8}$" or $3\frac{1}{2}$" (F) 5"
   (G) $5\frac{5}{8}$"

2. a. $\frac{7}{8}$" d. $1\frac{1}{2}$" or $1\frac{5}{8}$"
   b. $2\frac{1}{8}$" c. $\frac{1}{2}$" or $\frac{5}{8}$" e. $\frac{5}{8}$"

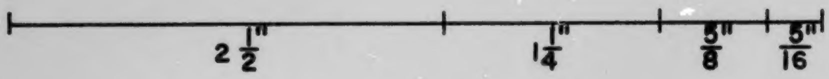
3. a.



b. 8

c. The divisions on the line picture the meaning of $5 \div \frac{5}{8}$; there are 8 pieces.

4. a.



b. $4\frac{11}{16}$" c. $4\frac{11}{16}$

5. a. $\frac{7}{16}$" c. $5\frac{1}{2}$" e. $\frac{3}{4}$" g. $\frac{1}{2}$"
   b. $1\frac{9}{16}$" d. $2\frac{1}{2}$" f. $3\frac{1}{2}$" h. $\frac{5}{8}$"

6. a. $AB \approx \frac{7}{16}$"

$BC \approx \frac{13}{16}$"

$CD \approx \frac{3}{16}$"

$DE \approx \frac{9}{16}$"

$EF \approx \frac{3}{4}$"

$FG \approx 1$"

$GH \approx \frac{1}{2}$"

$HI \approx \frac{2}{8}$"

$IJ \approx \frac{5}{8}$"

b. There is a possibility of a slight ($\frac{1}{16}$") discrepancy due to the approximate nature of measures.

*7.

Base 10	Base 2
$\frac{1}{2}$	$\frac{1}{10}$
$\frac{1}{4}$	$\frac{1}{100}$
$\frac{1}{8}$	$\frac{1}{1000}$
$\frac{1}{16}$	$\frac{1}{10,000}$

*8. Since all divisions are obtained by dividing existing divisions by 2, the number of sections increases by powers of 2, or, in base 2 notation, by adjoining a zero to show the number of divisions.

Answers to Exercises 11-2d

1. a. 48 in.
   b. 9 yd.
   c. 81 ft.
   d. 300 ft.
   e. 40 yd.
   f. 6 ft.
   g. 21,120 ft.
   h. $4\frac{1}{2}$ ft.
   i. $1\frac{2}{3}$ yd.
   j. $\frac{3}{4}$ yd.

2. 1760 yds.

3. Answers will vary.

4. Classroom, halls and other lengths can also be measured. It would be wise for the teacher to have a good idea of the measures assigned.

5. a. $7\frac{1}{2}$ in.
   b. 7 in.
   c. $8\frac{3}{4}$ in.
   d. $6\frac{3}{4}$ in.
   e. $7\frac{13}{16}$ in.
   f. $7\frac{3}{4}$ in.

   There may be some slight variation in above lengths.

6. a. Length $\approx$ 5400 in. (Answers in feet are acceptable. See part b.)
   Width $\approx$ 900 in.
   Height $\approx$ 540 in.

   b. Length $\approx$ 450 ft. $\approx$ 150 yd.
   Width $\approx$ 75 ft. $\approx$ 25 yd.
   Height $\approx$ 45 ft. $\approx$ 15 yd.

11-3. Precision and Error.

The teacher should reread the commentary for Section 11-1 before starting this section so as to have clearly in mind the only kind of error being considered in this section. Attempts to explain other types of error would probably lend to great confusion. Students may find this material difficult and the teacher is cautioned not to spend too much time on it.

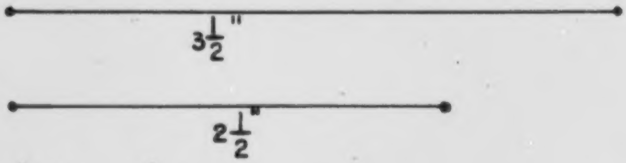
A discussion of "greater precision" has been deleted as it may be confusing. We would speak of measurements made with $\frac{1}{8}$-inch precision as having "greater precision" than those made with $\frac{1}{4}$-inch precision. We would mean that the precision is greater in the sense that the measurements are better, that is, that the error is smaller. Of two measurements with the same units, but with differently marked rulers, the one having "greater precision" is the one for which the reciprocal of the number used in designating the precision is greater. It is recommended that this difficult idea should not be touched.

Answers to Exercises 11-3a

1. 4
2. $3\frac{1}{2}$
3. $3\frac{3}{4}$
4. $3\frac{5}{8}$
5. 4
6. $4\frac{0}{2}$
7. $4\frac{0}{4}$
8. $4\frac{1}{8}$

Answers to Exercises 11-3b

1.



2. $3\frac{1}{2}$ in.; $2\frac{1}{2}$ in.

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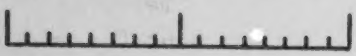
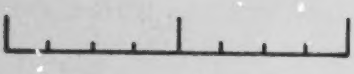
3. $\frac{1}{2}$ inch

4. $2\frac{3}{4}$ in.; $2\frac{1}{4}$ in.

5. $2\frac{3}{4}$ in.; $2\frac{1}{4}$ in.

6. $\frac{1}{4}$ inch

7.  (Divided into 8ths of an inch)

8.  (Divided into 4ths of an inch)

9.

Figure		(1)		(2)		(3)	
		$\frac{1}{2}$" marks	Error	$\frac{1}{8}$" marks	Error	$\frac{1}{16}$" marks	Error
(a)	ℓ	3"	$\frac{1}{4}$"	$2\frac{3}{4}$" or $2\frac{7}{8}$"	$\frac{1}{16}$"	$2\frac{13}{16}$"	$\frac{1}{32}$"
	w	$1\frac{1}{2}$"		$1\frac{3}{8}$"		$1\frac{3}{8}$" or $1\frac{6}{16}$"	
(b)	ℓ	2"	$\frac{1}{4}$"	2" or $2\frac{1}{8}$"	$\frac{1}{16}$"	$2\frac{1}{16}$"	$\frac{1}{32}$"
	w	2"		2" or $2\frac{1}{8}$"		$2\frac{1}{16}$"	
(c)	ℓ	$1\frac{1}{2}$" or 2"	$\frac{1}{4}$"	$1\frac{3}{4}$" or $1\frac{6}{8}$"	$\frac{1}{16}$"	$1\frac{3}{4}$" or $1\frac{12}{16}$"	$\frac{1}{32}$"
	w	1"		$\frac{7}{8}$" or 1"		$\frac{15}{16}$"	
(d)	ℓ	$2\frac{1}{2}$"	$\frac{1}{4}$"	$2\frac{5}{8}$"	$\frac{1}{16}$"	$2\frac{5}{8}$" or $2\frac{10}{16}$"	$\frac{1}{32}$"
	w	$\frac{1}{2}$"		$\frac{1}{2}$" or $\frac{5}{8}$"		$\frac{9}{16}$"	

11-4. Perimeter and Rectangles.

In this short section four terms are defined: perimeter, polygon, side, rectangle. The reason for the choice of the rather vague definition of a rectangle as a four-sided polygon with all corners "square" is that the terms "right angle" and "90-degree angle" are not available until Chapter 13.

The teacher should note that in the discussion of perimeter of a rectangle, the letters ℓ, w, and p denote the measures of lengths and not the lengths themselves. That is, if the lengths of the sides of a rectangle are 3 inches and 4 inches, then ℓ, w, p are respectively 3, 4, 14 not 3 inches, 4 inches and 14 inches. The reason for this is that we wish to confine addition and multiplication to operate on numbers and not on inches or oranges. No section on denominate numbers has been included. Once the measure of the perimeter has been found, the perimeter itself is found by appending the unit. That is, in the above example $p = 14$, the unit was the inch so that the perimeter is 14 inches.

Answers to Exercises 11-4a

1. a. A b. E,G c. H,I,J,K d. M,P e. none
2. HIJK
3. a. $3\frac{7}{8}$ inches c. $3\frac{3}{4}$ inches e. $5\frac{1}{4}$ inches
   b. $4\frac{3}{4}$ inches d. $3\frac{3}{8}$ inches
4. $\overline{IJ}$ opposite to $\overline{HK}$, $\overline{KJ}$ opposite to $\overline{HI}$
   The opposite sides have the same length.
5. The opposite sides are congruent.
6. A property of rectangles is that opposite sides are congruent. This is a property of all parallelograms of which rectangles are just a special subset.

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Answers to Exercises 11-4b

1. 1200 ft.

2. $2000

3. 50 ft. or $16\frac{2}{3}$ yds.

4. 38 ft. It does not matter where the doorways are located.

5. a. 12 ft. by 12 ft. - no; there is not enough fence.

   b. 8 ft. by 3 ft. - no; all the fence is not used.

   c. 8 ft. by 4 ft. - yes.

   d. Any of the following lengths and widths will work:

      11 ft. by 1 ft.
      9 ft. by 3 ft.
      7 ft. by 5 ft.
      10 ft. by 2 ft.
      8 ft. by 4 ft.
      6 ft. by 6 ft.

6. 700 inches; $19\frac{4}{9}$ yds.

7. a. $2\frac{3}{4}$ miles.

   b. Cost ≈ $687.50

   c. $1\frac{1}{4}$ miles

   He saved nothing in distance.

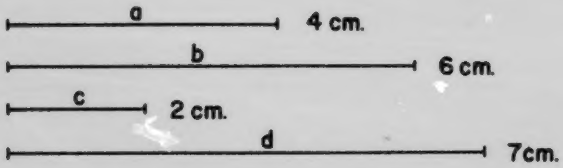
8. The side opposite the given side is 40 ft. long; the other two sides are each 80 ft. long.

11-5. Metric Units of Length.

The metric system is treated in this chapter principally as another group of measuring units which have the advantage of a decimal relationship. There sould be no attempt to teach the system as a whole. Only three units, the meter, centimeter, and millimeter, are introduced. Pupils should have an opportunity to study a meter stick to visualize the relationship between these units, so that they develop a mental picture of each. The two sets of exercises introduce the names of the units slowly and provide practice in measuring segments in centimeters and millimeters.

Answers to Exercises 11-5a

1.



2. Length $\approx$ 28 cm. $\approx$ 27.9 cm.
Width $\approx$ 22 cm. $\approx$ 21.6 cm.
Some variation should be permitted in the .1 cm. measurements.

3. AB $\approx$ 11 cm. $\approx$ 10.6 cm.
BC $\approx$ 7 cm. $\approx$ 6.9 cm.
AC $\approx$ 15 cm. $\approx$ 15.3 cm.

Totals $\approx$ 33 cm. $\approx$ 32.8 cm.

DE $\approx$ 13 cm. $\approx$ 13.3 cm.
EF $\approx$ 5 cm. $\approx$ 4.6 cm.
FG $\approx$ 8 cm. $\approx$ 7.7 cm.
DG $\approx$ 7 cm. $\approx$ 6.7 cm.

Totals $\approx$ 34 cm. $\approx$ 33.4 cm.

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4. a. 200 cm. e. 60 cm.
b. 700 cm. f. 5 cm.
c. 650 cm. g. 32 cm.
d. 120 cm. h. 128 cm.

5. a. 3 m. e. .5 m.
b. 7 m. f. 4.50 m.
c. 2.56 m. g. .75 m.
d. 1.85 m. h. .08 m.

Answers to Exercises 11-5c

1. AB ≈ 106 mm. DE ≈ 133 mm.
BC ≈ 69 mm. EF ≈ 46 mm.
AC ≈ 153 mm. FG ≈ 77 mm.
DG ≈ 67 mm.

2. a. 30 mm. e. 1150 mm.
b. 120 mm. f. 174 mm.
c. 28 mm. g. 1000 mm.
d. 63 mm. h. 3500 mm.

3. a. 4 cm. e. 15.6 cm.
b. 10 cm. f. 200 cm.
c. .1 m. g. 2 m.
d. 3.2 cm. h. 20.4 cm.

11-7. Chapter Review.

Answers to Exercises 11-7

1. inch, centimeter, meter, foot, mile, yard, etc.
2. cup, acre, quart, square foot, cubic inch, etc.
3. $8\frac{5}{8}$, $8\frac{7}{8}$.
4. a. 52' b. 12" c. $60\frac{1}{8}$"
5. $\frac{1}{8}$"
6. $\frac{1}{4}$"
7. $1\frac{3}{4}$ units
8. $\frac{3}{4}$ units
9. $1\frac{1}{4}$ units
10. AC ≈ 67 mm. ≈ $2\frac{5}{8}$"

    AB ≈ 73 mm. ≈ $2\frac{7}{8}$"

    BC ≈ 52 mm. ≈ $2\frac{1}{16}$"

Perimeter ≈ 192mm. or 19.2 cm. or $7\frac{9}{16}$".

11-8. Cumulative Review

Answers to Exercises 11-8

1. a. $9 \cdot 3 + 9 \cdot 2$
   b. $7 \cdot 6 + 11 \cdot 6$

2. a. Yes
   b. No

3. a. $6 < 8$
   b. $\frac{1}{2} < \frac{9}{17}$
   c. $\frac{9}{14} < \frac{20}{31}$
   d. $\frac{13}{15} > \frac{5}{6}$
   e. $\frac{3}{4} > \frac{25}{40}$
   f. $\frac{15}{18} < \frac{28}{32}$

4. Could be any base greater than 5.

5. 37_{eight}

6. a. 3: CDA, CDB, ADB,
   b. 2: CDA, ABC
   c. Point A
   d. If 2 different lines intersect, one and only one plane contains both lines.

7. a. $\frac{14}{16}$
   b. $1\frac{3}{16}$
   c. $\frac{12}{16}$
   d. $2\frac{8}{16}$
   e. $3\frac{0}{16}$
   f. $1\frac{3}{16}$

8. a. $10\frac{2}{5}$
   b. $3\frac{11}{12}$
   c. $3\frac{19}{24}$
   d. $2\frac{1}{2}$
   e. $2\frac{1}{3}$
   f. 1

9. $66 = 2 \cdot 3 \cdot 11$

10. The number 47 is a composite number.

11. 110 ft.

12. 20 percent

Sample Test Questions for Chapter 11

I. True or False

(F) 1. The counting numbers are all that are needed for both counting and measuring.

(T) 2. All measuring units can be subdivided.

(T) 3. A ruler is one form of a number line.

(F) 4. The proportion

$$\frac{1}{12} = \frac{576}{n}$$

could be used to find the number of feet in 576 in.

(T) 5. A measuring unit may have any size we choose.

(T) 6. Base 2 is related to the subdivisions of an inch on the ruler commonly used in school.

(F) 7. A centimeter is 100 times as long as a meter.

(T) 8. The length of a simple closed curve is called its perimeter.

(T) 9. A rectangle is a simple closed curve.

(T) 10. The smaller the unit, the more precise the measurement.

(F) 11. The more precise the measurement, the greater is the possible error.

(T) 12. A measurement of 7 miles has the same greatest possible error as measurement of 18 miles.

(F) 13. The term "greatest possible error" of a measurement refers to a mistake made in the measurement.

II. Multiple Choice

(c) 1. To measure a line segment, you must use as a unit:

(a) A #$2\frac{1}{2}$ tin can (d) a square inch
(b) A quart (e) None of these
(c) A line segment

(a) 2. Choose the best way to complete the statement, "standard units of measures are used because:

(a) It is important for people to use the same unit in dealing with each other."

(b) Standard units give more accurate measurements than units which are not standard."

(c) People have always used them."

(d) They all fit base 10 numerals very well."

(e) None of these

(b) 3. The markings on a ruler divide each inch into 8 equal parts. The correct way to report one measurement made with this ruler is:

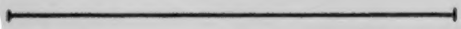
(a) $3\frac{4}{16}$ in.

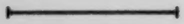
(b) $3\frac{2}{8}$ in.

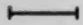
(c) $3\frac{1}{4}$ in.

(d) None of these

(c) 4. Which one of the following line segments is approximately 1 centimeter long?

(a) 

(b) 

(c) 

(d)

(e) None of these

(c) 5. If a rectangle is 3 ft. long and 5 ft. wide, its perimeter is:

a. 15 ft. b. 8 ft. c. 16 ft. d. 15 sq. ft.
e. None of these

(a) 6. If a line segment is measured to the nearest $\frac{1}{8}$ of an inch, the precision of the measurement is:

a. $\frac{1}{8}$ in. b. $\frac{1}{16}$ in. c. $\frac{1}{4}$ in. d. 1 in.
e. None of these

(c) 7. If a line segment has the measurement $\frac{3}{4}$ in. and the ruler measures to the nearest fourth of an inch, the line segment has a true length

a) between $\frac{1}{2}$ inch and 1 inch
b) between $\frac{5}{8}$ inch and $\frac{3}{4}$ inch
c) between $\frac{5}{8}$ inch and $\frac{7}{8}$ inch
d) between $\frac{2}{4}$ inch and $\frac{4}{4}$ inch
e) None of these.

III. Completion

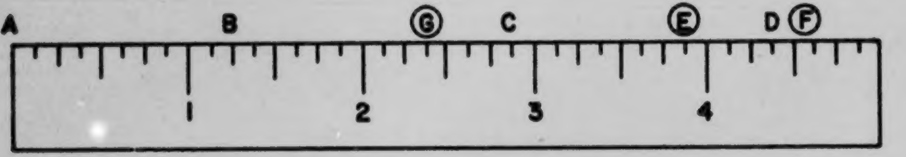
1. Measure these segments to the nearest 16th inch. Report the results so that the correct precision is indicated.

$3\frac{4}{16}$" (a) ______________________________

$4\frac{0}{16}$" (b) ______________________________________

$1\frac{8}{16}$" (c) ______________

2.



What number on the scale corresponds to:

$1\frac{1}{4}$" (a) Point B

$2\frac{7}{8}$" (b) Point C

$4\frac{3}{8}$" (c) Point D

3. On the number scale above mark the point which corresponds to the numbers given below and label them with the letter indicated.

(a) $3\frac{7}{8}$" E

(b) $4\frac{9}{16}$" F

(c) $2\frac{3}{8}$" G

4. Change these measurements to the other units.

(a) 4 ft.(a) 48 in. = ____ ft. (d) 54 in. = ____ yd.

(d) $1\frac{1}{2}$ yd.

(b) 3.7 cm.(b) 37 mm. = _____ cm. (e) 18 in. = ______ ft.

(e) $1\frac{1}{2}$ ft.

(c) 36 ft.(c) 12 yd. = ______ ft. (f) 4M = _______ cm.

(f) 400 cm.

Chapter 12

AREA AND VOLUME

Introduction.

The work in this chapter is a natural outgrowth of the work just completed on measurement. The student should find it relatively simple to extend his thinking from linear units of measure to both square units and cubic units of measure. The measurement of two and three dimensional figures is just as essential as the measurement of one dimensional figures in the development of certain geometric concepts and their applications. In this chapter, then, attention is focused upon development of basic concepts which underlie measurements of surfaces and solids.

Sufficient exercises on each new concept have been given so that the student will have ample practice with the new ideas and will gain confidence in his own ability to succeed.

12-1. Meaning of Area.

This section is designed to develop the concepts of area and the measurement of area. It is important that students understand that the area of a closed region is measured by using another closed region as a unit. Theoretically, any closed region may be chosen as the unit of measure and this idea is presented in the development and in the exercise material.

Answers to Class Exercises 12-1

1. This question is to be answered by observation.
    - a. $A > B$
    - b. $B < D$
    - c. $C > B$
    - d. $D > A$
    - e. $C < A$

2. In this exercise, the closed region A is taken as the unit of area. A will fit into B approximately 3 times, into C approximately twice, into D approximately 4 times, into E approximately 6 times. It will be impossible to fit A into F but the student will be able to see intuitively that F is the figure of largest area. It should be impressed upon the student that it is not always convenient to use the fitting process.

The order of size is F, E, D. B, C.

3. In this exercise attention is focused upon the idea of actual measurement of area. Each of the figures B, C, D, and E can be conveniently measured by R.

The answers are:

B = 2 units
C = 4 units
D = 5 units
E = 5 units

4. This exercise is intended to give the student some insight into the reason for the choice of the square as a unit of measure of area. It is desirable to stimulate discussion on the merits and disadvantages of each closed region as a unit of area. For example, the circle leaves uncovered sections in the interior of the room. The hexagon (or tile) is difficult to fit along the edges of the room. The choice between the isosceles right triangle and the square is essentially based on convenience.

5. The intention here is to provide additional practice in area measurement.

a. DEF = 1 unit
ABC = 4 units
DECF = 2 units
ADEC = 3 units

b. The regions DECF and BEFD.

12-2. Cutting Units of Area.

The aim of this section is to give the student an intuitive idea of the method of finding an area when the fitting procedure leaves portions of the area uncovered. The physical act of cutting up a unit of area into pieces which may be used to cover bits of surface has its counterpart in a corresponding calculation introduced in the next section.

Answers to Class Exercises 12-2

1. The answers will vary. This exercise provides the teacher with an opportunity to show the students that the number of units of area in a given surface depends upon the size of the unit used.

2. The area is a little less than 12 units. In this case, the students will have to cut out a number of the smaller units (at least 12) if the surface is to be covered. The teacher should permit the student to complete this exercise without interference. The student can cut out additional units if his first estimate is inadequate. We must use all the pieces of one unit before starting to cut the next one so that we can keep accurate count of the units used.

12-3. Area of a Rectangle.

In this section, the formula for the area of a rectangle is developed. The student should be ready to accept the square as the unit of area since he has had some experience with other possible units and their limitations. The definition of rectangle, at this point, must be informal because the work on angles comes in a later chapter.

The two methods of finding the area in the text illustrate the commutative property. The square of unit length is chosen for convenience; it makes computation simpler. The square with side 2 units or $1\frac{1}{2}$ units could be used but the computation involved in finding the area would be complex. It might be instructive for pupils to find the area of the rectangle in the text using the units of $2\frac{1}{2}$" and $1\frac{1}{2}$". In the work in this section it should be emphasized that the basic method of finding area is by counting. The counting, however, may usually be simplified by finding the product of the number of units in the length and the number of units in the width. Other units of area are the square foot and the square yard. These are analyzed in later work in the chapter. The property of rational numbers illustrated in this section is the commutative property.

Answers to Exercises 12-3a

1. b. 12
   - c. $\frac{1}{2}$" by $\frac{1}{2}$"
   - d. 15, $\frac{15}{4}$ square inches
   - e. $15\frac{3}{4}$ square inches
   - g. $(4\frac{1}{2})(3\frac{1}{2}) = (\frac{9}{2})(\frac{7}{2}) = \frac{63}{4} = 15\frac{3}{4}$ square inches

     This answer checks with the answers in parts d and e.
2. b. 6
   - c. $\frac{1}{4}$" by $\frac{1}{4}$"
   - d. 30, $\frac{30}{16} = \frac{15}{8}$ square inches
   - e. $7\frac{7}{8}$ square inches
   - f. $(3\frac{1}{2})(2\frac{1}{4}) = (\frac{7}{2})(\frac{9}{4}) = \frac{63}{8} = 7\frac{7}{8}$ square inches

*3. b. 20
   - c. $\frac{1}{3}$" by $\frac{1}{2}$"
   - d. 34, $\frac{34}{6}$ square inches
   - e. $25\frac{2}{3}$ square inches
   - f. $(5\frac{1}{2})(4\frac{2}{3}) = (\frac{11}{2})(\frac{14}{3}) = 25\frac{2}{3}$ square inches

Answers to Exercises 12-3b

1. d. 144

2. d. 9

3. c. The square 3 inches on a side by 6 square inches

4. c. An infinite number

5. a. 108 square feet
   b. 12 square yards

6. 8100 square feet, 900 square yards

7. Area is 3600 square feet. This is less than half the area of a baseball diamond.

8. a. 63 square inches
   b. 69 square inches
   c. $24\frac{3}{4}$ square inches
   d. $50\frac{3}{8}$ square inches
   e. $34\frac{11}{16}$ square inches

9. a. 100 square millimeters
   b. 1 square centimeter

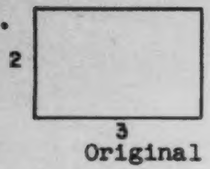
10. a. 5 cm.
    b. 2 cm.
    c. 10 square centimeters

11. a. 50 mm.
    b. 20 mm.
    c. 1000 sq. mm.

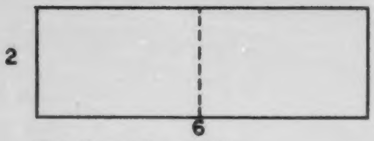
12. a. 6 square units
    b. 18 square units
    c. 24 square units
    d. Yes
    e. $3 \cdot 2 + 3 \cdot 6 = 3 \cdot (2 + 6)$

13. a. The larger one has twice the area of the smaller.

b.

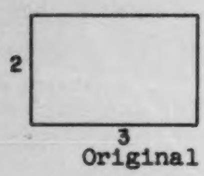


Original
Rectangle

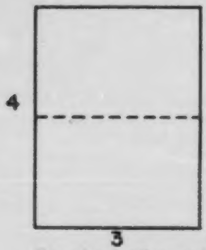


Rectangle with
Doubled Length

c.



Original



Rectangle with
Doubled Width

14. No
   a. Doubling the length of a rectangle (without changing the width of the rectangle) doubles the area.
   b. Doubling the width of a rectangle (without changing the length of the rectangle) doubles the area.

15. b. Doubling both the length and the width of a rectangle multiplies the area of the rectangle by 4.

16. 2 to 1

*17. 128 square inches

12-4. Approximation.

Although it is customary to treat measures as though they were exact in order to avoid complications it is advisable to call attention to the fact that all measures are approximate. If linear measures are approximate then it must be expected that areas obtained by multiplying linear measures will also be approximate. In this section, the upper and lower figures for the area are not computed but the student should be helped to realize that the number designating the area is approximate because it is obtained by finding the product of two approximate numbers.

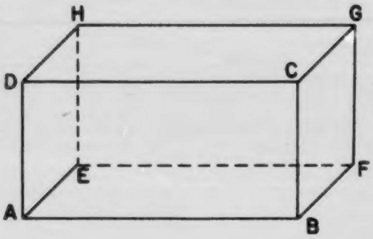
Answers to Exercises 12-4

2. a. Area $\approx$ $17\frac{1}{2}$ sq. ft., $1\frac{17}{18}$ sq. yd.
   b. Perimeter $\approx$ 17 ft., $5\frac{2}{3}$ yd.
3. a. Area $\approx$ 396 sq. ft.
   b. Cost $\approx$ $237.60
4. Area $\approx$ $\frac{3}{25}$ sq. mi.
5. Approximately 14 pounds
6. a. Area $\approx$ 3240 sq. in.
   b. Area $\approx$ $22\frac{1}{2}$ sq. ft.
7. Area $\approx$ 171 sq. ft.
8. a. Floor area $\approx$ 570 sq. ft.
   b. It does not matter where the opening is.
9. a. $7\frac{1}{2}$ inches
   b. $9\frac{1}{2}$ inches
   c. $8\frac{1}{2}$ inches
   d. $10\frac{1}{2}$ inches
   e. $71\frac{1}{4}$ square inches
   f. $89\frac{1}{4}$ square inches

12-5. Rectangular Prism.

Just as a rectangle is composed of the points on its segment, so a rectangular prism is made up of the points on its faces, i.e. on its surface. Thus if a brick is suggested as an example of this figure, point out that the prism consists only of the surfac of the brick. For this reason we do not speak of the volume of a rectangular prism, but the volume of a rectangular solid, since we define a rectangular solid as the set of points consisting of a rectangular prism and its interior. When we speak of the area of a prism we will be speaking of the area of the faces of the prism, which is the surface area. As in the previous sections, the stating of a measurement requires both the number and the unit used, but any letter in a number sentence stands only for a number. A rectangular solid has 3 pairs of opposite faces. Each face is a rectangular region.

Models and illustrations are necessary to help pupils visualize a rectangular prism with its 6 faces, 12 edges, and 8 vertices. The approach is frankly intuitive. However, once it has been agreed that the faces are rectangles, this can be used to deduce the fact that opposite faces have the same measurements. For example in the following figure:



$\overline{EF}$ and $\overline{AB}$ have the same length because they are opposite sides of rectangle ABFE.

$\overline{AB}$ and $\overline{CD}$ have the same length because they are opposite sides of rectangle ABCD.

$\overline{CD}$ and $\overline{GH}$ have the same length because they are opposite sides of rectangle DCGH.

Thus the four segments $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, $\overline{GH}$ all have the same length. Similarly $\overline{AE}$, $\overline{DH}$, $\overline{CG}$, and $\overline{BF}$ have the same length and $\overline{AD}$, $\overline{EH}$, $\overline{FG}$, and $\overline{BC}$ have the same length. This shows that any two opposite faces have the same measurements.

The intersection of two adjacent faces is a line. The intersection of three faces is a point.

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Answers to Exercises 12-5

These exercises will probably be difficult, not because of the arithmetic, which is easy, but because of the spacial visualizing required. It is suggested that in doing these problems each pupil keep a box or other model in front of him to help him visualize the problems that are presented. Except for the difficulty of visualizing, the problems are applications of work on rectangles in the last section. There is further drill on change of units.

In discussing the choice of a unit of volume bring out that its edges are units of length and its faces units of area, while its interior is the unit of volume.

1. ABCD and EFGH, or ADFE and BCGH, or ABHE and CDFG.
2. ABCD and DCGF, or ABCD and ABHE, or BCGH and DCGF, or ADFE and ABHE, etc:
3. a. $\overline{AB}$, $\overline{DC}$, $\overline{FG}$, $\overline{EH}$, $\overline{AD}$, $\overline{DF}$, $\overline{EF}$, $\overline{AE}$, $\overline{BC}$, $\overline{CG}$, $\overline{HG}$, $\overline{BH}$ are all edges.
   b. There are 12 edges on a rectangular prism.
4. $\{\overline{AB}, \overline{DC}\}$, or $\{\overline{AD}, \overline{EF}\}$, or $\{\overline{AE}, \overline{BH}\}$ etc.
5. a. A, B, C, D, E, F, G, H, are all vertices.
   b. There are 8 vertices on a rectangular prism.
6. $\overline{AB} = 2$, $\overline{EH} = 2$, $\overline{FG} = 2$, $\overline{DF} = 4$, $\overline{AE} = 4$, $\overline{BH} = 4$, $\overline{AD} = 3$, $\overline{EF} = 3$, $\overline{HG} = 3$.

12-6. Area of a Prism.

The area of a prism is the sum of the areas of the six faces. It is a simple matter to demonstrate the reasoning used to construct the formula for the area of a prism by finding the areas of the faces and finding the sum of these areas.

Answers to Exercises 12-6

1. 166 square inches
2. 152 square inches

3. 230 square feet, $25\frac{5}{9}$ square yards

4. The area of the piece of glass is 200 sq. in., or $1\frac{7}{18}$ sq. ft. The area of the wooden box is 1100 sq. in., $7\frac{23}{36}$ sq. ft.

5. a. 180 tiles
   b. 720 tiles

6. 396 square feet. It does not matter where the windows are placed.

7. 3 quarts of paint

8. 312 inches, 26 feet, $8\frac{2}{3}$ yards

9. a. a square
   b. 1944 square inches of wood
   c. $13\frac{1}{2}$ square feet

*10. $S = \ell w + 2hw + 2\ell h$

12-7. Meaning of Volume.

The concept of volume and the sizes of different units of volume need much reinforcing. Models of cubic inches will be constructed but models of cubic feet and cubic yards are necessary also and practice in estimating volumes is helpful. A framework for a model of a cubic yard can be formed from 12 yardsticks. Most adults are astounded at the size of a cubic yard when they see one. One teacher had students bring in cardboard cartons that could be cut to the size of a cubic foot, and then assembled a model cubic yard from 27 of these boxes. Grocery cartons which are partitioned by cardboard dividers may help pupils visualize the subdivision of a volume into units.

Answers to Exercises 12-7

Except for Problem 2, the problems of the set should be done wholly in class. It is extremely important that the pupil see and be able to visualize that the effect of doubling the length of a rectangular prism amounts to laying two prisms end to end, both just like the original one. Similarly, doubling two of the measurements amounts to putting four such prisms together and doubling all three measurements is equivalent to putting eight such prisms together. It is this visual perception, not a counting up of the cubes used, that is really significant. Also the models formed here show nicely the layers of cubes which are used in the next development.

1. Cubic foot - volume of a cube each edge of which is 1 ft.
   Cubic yard - volume of a cube each edge of which is 1 yd.

3. 24

4. The number of cubes for each of the new solids created in parts (b), (c), and (d) is just double that in the original solid.

5. a. Doubling one measurement (length, width, or height) of any rectangular prism doubles the volume of the solid.
   b. The ratio is $\frac{2}{1}$.

6. b. Ratio of the number of cubes in the new prism to the number in the original prism is $\frac{4}{1}$. That is, there are 4 times as many.
   c. The result would be the same if any two of the measurements are doubled.

7. Doubling any two measurements of any rectangular prism quadruples the volume of the solid. (i.e. multiplies it by 4)

8. b. There are eight times as many cubes in the new figure. That is, the ratio is $\frac{8}{1}$.

9. a. Doubling all three measurements of any rectangular prism yields a prism whose volume is 8 times that of the original solid. The ratio is $\frac{8}{1}$.

12-8. Volume of a Rectangular Solid.

If the student has gained the visual concepts with which he worked in the last section, he should readily see that the volume of any rectangular solid may be found merely by multiplying the area of the base by the height. Once this idea is established, it should be relatively easy to see that the volume may also be found by multiplying the measures of the length, width, and height because the base is a rectangle. Emphasis should be placed on the fact that any of the faces can be used as the base of a rectangular solid. Here is an opportunity for the teacher to review the commutative and associative properties of rational numbers as is indicated in the paragraph preceding Exercises 12-8b.

Answers to Exercises 12-8a

These problems have all been designed to emphasize the relation symbolized by the number sentence $V = Bh$. They have been deliberately chosen in such a way that the actual shape of the base is not known in any of them. This is to lay the foundation for the discussion later of volumes of prisms and cylinders, as well as the specific case of the rectangular prism which is discussed next.

1. 85 cu. ft. (approx.)
2. 20 cu. ft. (approx.)
3. a. The number of cubic units of volume in a rectangular solid is the product of the number of square units of area in its base and the number of linear units in its height.
   b. $V = Bh$
4. 2 ft. (approx.)
5. 20 sq. ft. (approx.)
6. Volume of the room is 1600 cubic feet. (approx.)
   30 children require 1500 feet, so that 30 children is a legal number.
   Greatest legal number of children is 32.

Answers to Exercises 12-8b

1. 24 cu. ft.

2. 1260 cu. in.

3. a. Area = 187 sq. in. b. Volume = 748 cu. in.

4. 9 cu. ft.

5. a. Volume = 81 cu. ft. b. 3 cu. yds.

6. The number of cubic units of volume in a rectangular prism is the product of the numbers of linear units in the length, width, and height.

7. $V = \ell wh$

8. a. The number of inches in each edge of a cubic foot is 12, so the volume is $V = 12 \cdot 12 \cdot 12 = 1728$, or 1728 cu. in.
   b. There are 27 cubic feet in a cubic yard.

9. a. The volume is 27 cubic inches.
   b. This volume is larger than 3 cubic inches.

10. $\frac{1}{2}$ inch.

11. b. 10,080 cubic feet.

12. b. Volume = 120 cu. in.
    c. The ratio of the volume of the new box to the old is $\frac{8}{1}$.

13. a. Volume = $\frac{27}{8}$ cu. in. or $3\frac{3}{8}$ cu. in.
    b. Area of one face = $\frac{9}{4}$ sq. in. or $2\frac{1}{4}$ square in.
    c. Surface area = $\frac{27}{2}$ sq. in. or $13\frac{1}{2}$ sq. in.

14. 3520 cu. ft.

Answers to Exercises 12-8c

1. Weight ≈ 6000 lbs. or 3 tons.

2. a. Volume ≈ 189,000 cu. ft. b. 7000 cu. yds.

3. Weight of gold is 2250 lbs. The men can lift 2000 lbs., so they could not lift the chest.

4. a. The tank contains approximately 2880 cu. in. of water.
   b. Approximately $12\frac{36}{77}$ gallons of water.

5. 8 cu. inches.

6. The number of lbs. of water $\approx 390\frac{5}{8}$.

7. 2 minutes.

8. 64 cubic inches.

9. Volume $\approx 183\frac{3}{4}$ cu. in.

*10. Depth $\approx 1\frac{1}{2}$ ft., or 18 in.

Answers to Exercises 12-8d

1. 154 sq. ft. of carpet. Volume $\approx$ 1386 cu. ft.

2. Volume of the pantry is 498 cu. ft. or $18\frac{4}{9}$ cu. yd.

3. Doubling each dimension makes the new volume 8 times the original. The motor should have 8 times the power of the original motor.

12-9. Dimension.

In the discussion of dimension some of your better pupils may raise the question of describing the location of the sugar in other ways than by motions parallel to the edges of a room. For example, in the second figure in the section under Dimension it may be suggested that the fly at A might simply point out the direction of S and tell his friend to crawl a certain distance in that direction. This is an excellent idea. However, note that these directions still call for two numbers, one describing the angle telling the direction in which the fly is to crawl, and the other giving the distance he must crawl. A precise definition of dimension involves very substantial difficulties beyond the scope of this course, but you will find that any "reasonable" way of describing location of points in the different sets will use the same number of numbers in the description, so that the concept of dimension has meaning.

A point has 0 dimensions.

Answers to Exercises 12-9

1.

a.	1 dim.	f.	3 dim.	k.	3 dim.
b.	2 dim.	g.	1 dim.	l.	3 dim.
c.	1 dim.	h.	2 dim.	m.	2 dim.
d.	2 dim.	i.	2 dim.	n.	3 dim.
e.	2 dim.	j.	2 dim.		

Answers to Exercises 12-10

1. a. $\frac{231}{4}$ cu. in. or $57\frac{3}{4}$ cu. in.
   b. Yes.
   c. Often there is a roof-shaped top containing the pouring spout but not filled with milk.
2. Less than a quart by $2\frac{59}{64}$ cu. in. (Volume of container is $54\frac{53}{64}$ cu. in.)
3. The volume may be the same but if the materials are different, the weights may be different. The old saying is roughly true for water.
4. a. $64\frac{111}{128}$ cu. in. or ≈ 65 cu. in. (Remember, however, that this result is an approximation. The form implies a precision which is not achieved.)
   b. $67\frac{3}{8}$ cu. in., or ≈ 67 cu. in.
   c. No.
5. a. $35\frac{5}{32}$ cu. in., or ≈ 35 cu. in.
   b. $33\frac{11}{16}$ cu. in., or ≈ 34 cu. in.
   c. The box holds about 1 cu. in. more than it should.
6. Presumably in measuring dry quantities such as berries and the like, there are air spaces not filled with anything. The problem is resolved by increasing the total volume which is to be called a quart.

7. \$1.07

8. a bushel costing \$3.50

9. Since 1 gal. = 231 cu. in., then $3\frac{1}{2} \times 3\frac{1}{2} \times h = \frac{231}{2}$.

*10. 3 in. by 7 in. by 11 in.

12-12. Chapter Review.

Answers to Exercises 12-12

1. a. 117 square feet
   b. $\frac{315}{8}$ square feet

2. a. 833 cubic inches
   b. $\frac{1309}{8}$ cubic feet

3. a. 3750 square feet
   b. 84 square feet
   c. 3666 square feet

4. a. $\frac{125}{8}$ cubic inches
   b. $\frac{75}{2}$ square inches

5. a. 6
   b. 8
   c. 12

6. 336 square inches

7. \$729

8. a. 178 square feet
   b. 140 cubic feet

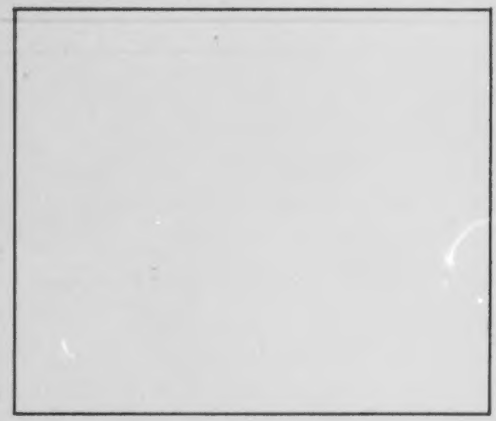
9. a. 2
   b. 1
   c. 3
   d. 2
   e. 1
   f. 2
   g. 3

10. $261\frac{9}{11}$ gallons

12-13. Cumulative Review.

Answers to Exercises 12-13

1. There are none. This is the empty set.
2. 0, 24, 48, 72, 96, 120, etc.
3. a. False
   b. True
   c. False
4. {1, 2, 5, 10, 25, 50}
5. a. XXIII
   b. XIX
6. {1, 2, 4}
7. 361.25 miles
8. a. $3 > 0$
   b. $13 < 15 < 17$
9. $216
10. a, b, and c
11. a. Surface Area is about 190 sq. in.
    b. $V \approx 150$. Volume is about 150 cu. in.
12. a.



    b. $A \approx 8\frac{15}{16}$. Area is about $8\frac{15}{16}$ sq. in.

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13. a. The set of whole numbers greater than 0 and less than 10.

    b. The family including both parents and children.

14. a. Point C　　c. $\underleftrightarrow{\overline{CD}}$

    b. $\overline{BF}$　　d. BF

    e. The empty set

15. a. $8\frac{1}{2}$ in.　　b. $8\frac{4}{8}$ in.　　c. $8\frac{8}{16}$ in.

Sample Test Questions for Chapter 12

Teachers should construct their own tests, using carefully selected items from those given here and from their own. There are too many questions here for one test. Careful attention should be given to difficulty of items and time required to complete the test.

Multiple Choice, Completion, and Matching.

1. A hall is 6 feet long and $2\frac{1}{2}$ feet wide. How many square yards of carpet will cover it?

    a. 17　　d. $5\frac{2}{3}$

    (b) b. $1\frac{2}{3}$

    c. 15　　e. None of these

2. The interior of a 4 inch cube is:

    a. The same as 4 cu. in.

    b. Smaller than 4 cu. in.

    (c) c. 16 times as large as 4 cu. in.

    d. 4 times as large as 4 cu. in.

    e. None of these

3. All rectangles with perimeters of 20 inches:

    a. Have the same interior measure.

    b. Have interior measures that increase as the base increases in length.

    c. Have interior measures that increase as the height increases in length.

(e) d. Have the same interior measure as a square with a 20 inch perimeter.
e. None of these.

4. 2 cu. ft. is equal to:
a. 24 cu. in.
(b) b. 3456 cu. in.
c. 266 cu. in.
d. 18 cu. yd.
e. None of these

5. Choose from the right-hand column, the term which best describes each term in the left-hand column and write its number on the line.

a.	Face of a cube (7)	1.	Ray
b.	Side of a rectangle (2)	2.	Line segment
c.	Side of an angle (1)	3.	Point
d.	Intersection of edges of a rectangular prism (3)	4.	Line
		5.	Plane
e.	Face of a rectangular prism (6)	6.	Rectangle
		7.	Square

6. A rectangular prism has (6) faces, (12) edges, and (8) vertices.

7. The interior of a rectangular prism is (3) dimensional while any one of its faces is (2) dimensional.

8. If the length and width of a rectangle are doubled, the interior of the new rectangle is (4) times that of the original one.

9. If the length and width of a rectangle are doubled, the perimeter of the new rectangle is (2) times that of the original one.

10. Which of the units in the right-hand column would be the best to use to measure the thing listed in the left-hand column? (A unit may be used more than once.) Write the number of the unit in the right-hand column on the line in the left-hand column.

a. Air space in a room (4)
b. Linoleum needed to cover a shelf (3)
c. Amount of water in a small aquarium (4) or (9)
d. A clothes line (2) or (7)
e. Space in a refrigerator (4)
f. An angle (1)

1. Degree
2. Foot
3. Square foot
4. Cubic foot
5. Mile
6. Square mile
7. Yard
8. Square inch
9. Cubic inch

11. The position of any point on a line can be described by (1) number(s). The position of any point on a plane surface can be described by (2) number(s). The position of any point interior to a solid can be described by (3) number(s).

12. Jim has 26 feet of left-over fencing to use around a small garden. Give two different sets of dimensions he could use. Find the area of the garden in each case. (Any two lengths that add to 13'. Areas will vary.)

13. A rectangular prism is 8 ft. long, 2 ft. wide and 3 ft. high. The area of the largest face is (24) sq. ft. The volume of the interior of the prism is (48) cu. ft.

14. An area is found to be 18 sq. ft. This is the same as (2592) sq. in. or (2) sq. yd.

15. a. Draw a rectangle $2\frac{3}{8}$ in. by 3 in.
b. Find the perimeter of this rectangle. ($10\frac{3}{4}$")
c. Find its area. ($7\frac{1}{8}$ sq. in.)

16. A sand box is 3 feet wide and 4 feet long. How many cubic feet of sand are needed to fill the box to a depth of 10 in.? (10 cu. ft.)

17. An aquarium is 14 inches wide, 22 inches long and holds 12 gallons of water. How deep is the water? (1 gal. = 231 cu. in.) (Answer: 9 in.)

*18. A rectangular playground is 180 ft. by 330 ft.

a. What is its area in square feet? In square yds.? (59,400 sq. ft. -- 6,600 sq. yd.)

b. What is its perimeter? (1020 ft.)

c. What would it cost to put blacktop on the playground at 90¢ a square yard? ($5,940)

d. A fence is to be put around the two short sides and one long side of the playground. What would this fence cost at $2.25 per foot? ($1552.50)

19. A chest is 30 inches wide, 2 feet high, and 5 feet long.

a. Find the area of the surface. (55 sq. ft.)

b. A small can of stain will cover 30 square feet. Is one small can enough to stain the top and sides? (No.)

Chapter 13

ANGLES AND PARALLELS

Introduction

This chapter is primarily concerned with classification and measurement of angles. The past two chapters have emphasized the need for standard units of measurement in finding lengths, areas, and volumes. Here we find a need for a standard unit to measure angles and introduce the protractor as a tool for such measurement.

Each youngster should obtain or be given a protractor, and adequate time should be spent to insure proficiency with this basic tool.

Laboratory-type experiments are introduced in this chapter and continued throughout Chapter 14. Students enjoy this type of experience provided emphasis is placed on having them "discover" relationships by their own observations. Do not reduce this interest by telling them the answers. Furthermore, do not let them jump to hasty conclusions without adequate observations.

13-1. Measurement of Angles.

In Chapter 7, the pupils have been introduced to the concept of angle as the set of points on two rays with a common endpoint. They have learned to describe the position of a point as being on the angle, in the interior of the angle, or in the exterior of the angle. The measurement of an angle follows essentially the same ideas as the measurement of a line segment; that is, (1) the unit for measuring an angle must be itself an angle; (2) the interior of the angle is subdivided by drawing rays which form angles like the unit angle; (3) the measure of the angle is the number of unit angles into which it is subdivided.

This section provides another opportunity to stress the concept of measurement and the need for standard units of measurement. Thus, time should be taken in class to discuss the use of an arbitrary unit angle in measuring an angle.

The standard unit, the degree, and the scale for measuring angles in degrees, are introduced. The unit angle is determined by a set of 180 rays which form 180 congruent angles, each of which is an angle of one degree. The scale is established by numbering the rays in order from zero to 180.

The most common type of protractor shows the scale only from 0 to 180, and usually contains two such scales, one with 0 at the right end and the other with 0 at the left end of the semicircular scale. Each pupil should have a protractor and should become proficient in using it, both to measure a given angle and to draw an angle of a specified number of degrees.

Students will need a great deal of practice in the use of a protractor if they are to become proficient in using it. It might prove wise for the teacher to prepare sheets with numerous angles whose measure the student is to estimate first and then find with a protractor. After some practice, the class should be able to use protractors to measure the angles of a triangle.

Stress the use of the word congruent; that is, angles with the same measure are called congruent angles.

Note the correct pronunciation cón-gruent.

Answers to Exercises 13-1a

1. $\angle$ CAB; $\angle$ BAC   2. $\overrightarrow{AC}$, $\overrightarrow{AB}$

3. The vertex of the angle.

4. P   5. R   6. C, A, or B

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Answers to Exercises 13-1b

1. 15
2. 35
3. 60
4. 90
5. 100
6. 25
7. 60
8. 145
9. 40
10. 85
11. 140
12. 65

Answers to Exercises 13-1c

1. $\angle A = 50^\circ$, $\angle B = 90^\circ$, $\angle C = 120^\circ$, $\angle D = 75^\circ$, $\angle E = 100^\circ$, $\angle F = 60^\circ$, $\angle G = 130^\circ$, $\angle H = 40^\circ$.
2. 35
4. 180

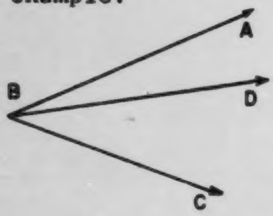
13-2. Two Lines in a Plane.

The concepts of this section can best be illustrated by using two objects, such as yardsticks or pointers to represent lines in space. Show that these lines can intersect, be parallel, or be skew. If they intersect, then they must lie in a plane. (See Chapter 4, this is a good place to review the terminology of non-metric geometry.)

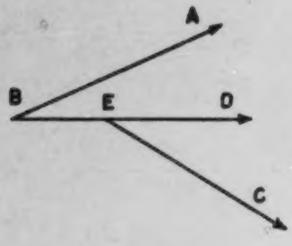
In the drawing, ℓ_1 and ℓ_2 determine four angles. The two angles which are not named in the text are angles EAB and DAC.

13-3. Adjacent Angles.

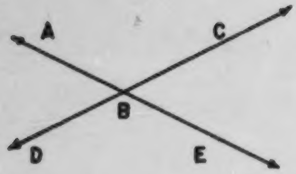
As you develop the concept of adjacent angles, it is wise to give examples which violate each of the listed requirements. For example:



$\angle ABD$ is not adjacent to $\angle ABC$;
Their interiors have points in common.



$\angle$ABD is not adjacent to $\angle$DEC;
They do not use the same point for a vertex.



$\angle$ABD is not adjacent to $\angle$CBE;
They do not have a common ray.

Answers to Exercises 13-3

1. In Figure 1: $\angle$EAD, $\angle$DAC, $\angle$CAB, $\angle$BAE.

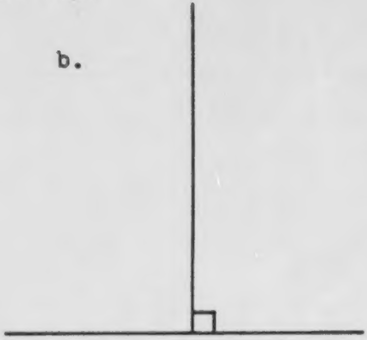
   In Figure 2: $\angle$YXR, $\angle$RXH, $\angle$HXW, $\angle$WXY.

2. c.

3. a. They do not have a common vertex or a common ray.
   b. They do not have a common ray.

4. a. $\angle 1$, $\angle 2$
   $\angle 2$, $\angle 3$
   $\angle 3$, $\angle 4$
   $\angle 4$, $\angle 5$
   $\angle 5$, $\angle 6$
   $\angle 6$, $\angle 1$

   b.



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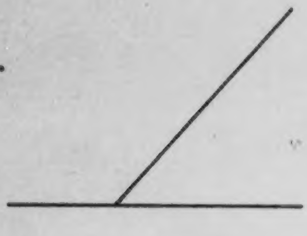
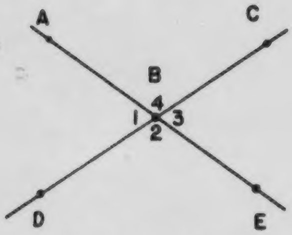
5. a.



b.



c. $\angle$ ABD, $\angle$ DBE; $\angle$ DBE, $\angle$ EBC; $\angle$ EBC, $\angle$ CBA;

$\angle$ CBA, $\angle$ ABD.

13-4. Vertical Angles.

It is important to spend some time to point out clearly the distinction between vertical and adjacent angles. A number of different ways to classify pairs of angles will be developed in this chapter and one must be careful that students do not become confused with the terminology introduced. The pairs of vertical angles in the figure are $\angle$ AED and $\angle$ BEC, $\angle$ AEB and $\angle$ DEC.

Answers to Exercises 13-4

1. b. 2. No. One of the requirements for vertical angles states that they cannot be adjacent angles.

3. $\angle$ COF, $\angle$ EOD

$\angle$ AOC, $\angle$ BOD

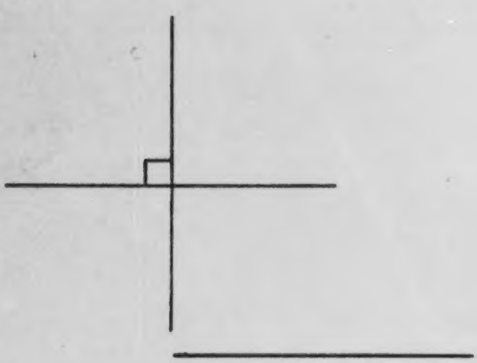
$\angle$ AOE, $\angle$ FOB

4. $\angle$ FOA, $\angle$ BOE

$\angle$ COE, $\angle$ FOD

$\angle$ AOD, $\angle$ COB

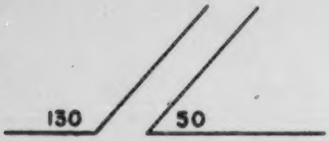
5.



13-5. Supplementary Angles.

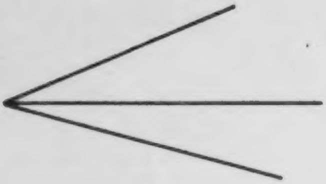
In developing the concept of supplementary angles be certain that the student recognizes that these angles need not be adjacent. Thus, in the figure, ∠BAC (30°) and ∠CAR (150°) would be supplementary even if they were not adjacent. It might be well to point out this distinction through the use of such figures as the following:

(a)



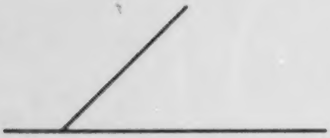
Supplementary angles which are not adjacent.

(b)



Adjacent angles which are not supplementary.

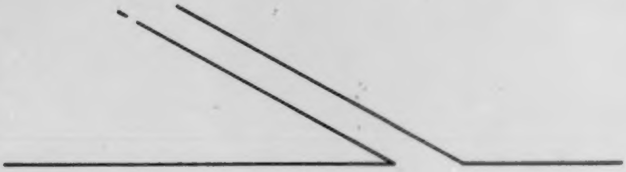
(c)



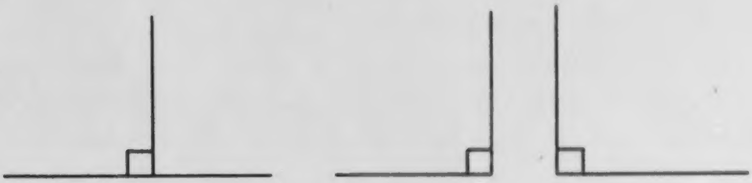
Supplementary adjacent angles.

Answers to Exercises 13-5

1. $m\angle A = 38$ $m\angle D = 120$
   $m\angle B = 90$ $m\angle E = 55$
   $m\angle C = 130$ $m\angle F = 95$
2. 180, yes, no.
   If the union of the interiors of two adjacent angles and their common half-line is a half-plane, then the angles are supplementary angles.
3. (a), (d), (e). (See answer to Problem 2)
4. $\angle CAB$, $\angle BAE$; $\angle BAE$, $\angle EAD$; $\angle EAD$, $\angle DAC$; $\angle DAC$, $\angle CAB$.
5.



6.



7. 12

13-6. More About Vertical Angles.

In this section, you will find the first of a number of experiments which will be presented in both this and the next chapter. Allow time for the students actually to perform the experiment in class and let them make their own discoveries if at all possible. Stress the fact that this is a laboratory experiment, which might be carried on in a science room, but which does not constitute a proof. (The concept of proof will be illustrated in the next chapter.)

Answers to Exercises 13-6

1. c., $\angle x$ and $\angle y$ are vertical angles.
2. a. 180
   b. 180
   c. by subtracting $m\angle y$ from 180.
   d. $m\angle x = 180 - m\angle y$
   $m\angle z = 180 - m\angle y$
   e. $m\angle x = m\angle z$

13-7. Right Angles.

Two lines are said to be perpendicular if they meet so as to form a pair of congruent adjacent angles. It is important here to review the meaning of the words "congruent" and "adjacent" so that this definition becomes clear. The term "perpendicular" is also applied to line segments and rays. However, as it is generally used in this chapter, the word "perpendicular" will refer to lines.

Angles are classified as acute, right, or obtuse, according to their measures. The student should gain practice in estimating the number of degrees in an angle by comparing it with a right angle. In Exercises 13-7c, it may be well to have the students estimate the measure of each angle given in Problem 1 before asking them to find the measure with a protractor.

Answers to Exercises 13-7a

(Class Discussion)

1. See Section 13-3: their interiors have no point in common, they use the same point for a vertex, they have a common side.
2. They are supplementary.
3. No. (See discussion in Teachers Commentary -- Section 13-5)
4. No.

Answers to Exercises 13-7b

1. a. Drawing 3
   b. Drawing 5 (3 and 1 could be so interpreted also)
   c. Drawing 2
   d. Drawing 1
   e. Drawing 6
   f. Drawing 4
2. a. 180; they are supplementary angles.
   b. equals; perpendicular lines form congruent angles, by definition.
   c. 90
   d. vertical; vertical.
   e. 90; vertical angles have equal measures.

Answers to Exercises 13-7c

1. a. The right angles are $\angle B$ and $\angle E$.
   b. The acute angles are $\angle A$ and $\angle C$.
   c. The obtuse angles are $\angle D$ and $\angle F$.
2. $\angle A = 27^\circ$, $\angle B = 90^\circ$, $\angle C = 37^\circ$, $\angle D = 100^\circ$, $\angle E = 90^\circ$, $\angle F = 109^\circ$.

3. a. 0, 90
   b. 90, 180

4. a. Obtuse angles: $\angle$ BAF, $\angle$ BAG, $\angle$ BAH

   Acute angles: $\angle$ BAC, $\angle$ BAD

   Right angles: $\angle$ BAE

   b. Acute angles: $\angle$ EAD, $\angle$ EAC, $\angle$ EAF, $\angle$ EAG, $\angle$ EAH

   Obtuse angles: none

   Right angles: $\angle$ EAB, $\angle$ EAK

   c. Right angles: $\angle$ KAE

   Obtuse angles: $\angle$ KAD, $\angle$ KAC

   Acute angles: $\angle$ KAH, $\angle$ KAG, $\angle$ KAF

5. a. $\angle$ ABC is acute $\angle$ JML is obtuse
   $\angle$ EDF is a right angle $\angle$ PRN is a right angle
   $\angle$ HKG is acute $\angle$ QTS is obtuse

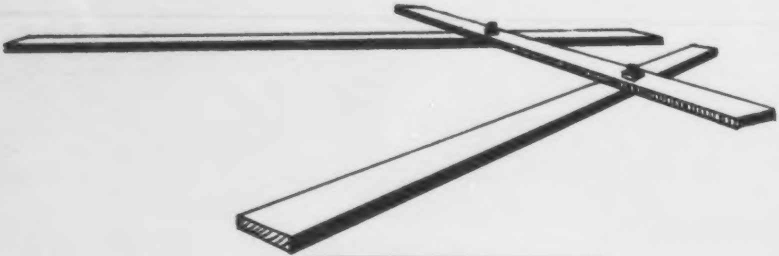
   b. See part (c)

   c. $\angle B = 10^\circ$, $\angle D = 90^\circ$, $\angle K = 65^\circ$, $\angle M = 115^\circ$,
   $\angle R = 90^\circ$, $\angle T = 130^\circ$.

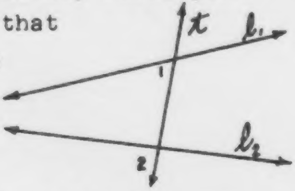
13-8. Three Lines in a Plane.

Review the concepts of Section 13-2 about two lines in a plane. Then, let the students discuss the different ways in which three lines in a plane may intersect.

It will prove helpful for the work of the remaining part of this chapter to make a demonstration model constructed of three flat sticks free to turn on pivoted connections as follows:



13-9. Corresponding Angles.



It is important, in this section, to identify corresponding angles clearly. In doing this, be sure that you present some examples in which the lines are not parallel. Thus, $\angle 1$ and $\angle 2$ are corresponding angles, but ℓ_1 and ℓ_2 are not parallel lines.

If an overhead projector is available, by all means use it for this section. On one piece of plastic film, draw (with china-marking pencil) line ℓ, point A on ℓ, and transversal t. On another piece, draw line m. By superimposing the second piece on the first and projecting the image on the chalkboard, line m can be rotated to different positions through Point A and observations as to angle measures and intersections can be tabulated on the chalkboard. Since the figure is projected on the chalkboard, marks may be erased without obliterating the figure.

Answers to Exercises 13-9

1. c.
2. $\angle a$, $\angle e$; $\angle c$, $\angle g$; $\angle b$, $\angle f$; $\angle d$, $\angle h$.
3. a. The intersection of the two angles is empty.
   b. Their interiors are not on the same side of the transversal.

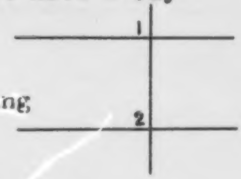
c. The intersection of the two angles is a segment not a ray.

d. Same as b.

e. The intersection of the two angles is not a ray on the transversal.

4. a. No, two corresponding angles do not have a ray in common.

   b. Yes, if they are right angles. Angles 1 and 2 are corresponding angles and each is a right angle.



5. No.

   Yes. $m\angle a = m\angle c = 80$ Vertical angles

   $m\angle d = 100$ $\angle a$ and $\angle d$ are supplementary and $m\angle d$ is that number which added to 80 gives 180.

   $m\angle d = m\angle e = 100$ Vertical angles

6. $m\angle a$ and $m\angle c$ are two names for the same number $m\angle b$.

7. a. Yes c. Yes

   b. Yes d. 6; 12; 12; 12.

   (ℓ_1, ℓ_2, and ℓ_3 are transversals because each line intersects the other two lines in distinct points.)

13-10. Parallel Lines and Corresponding Angles.

Once again emphasis should be placed on experimentation as a means of obtaining statements which appear to be true. This does not, however, constitute a proof.

A major concept to be developed in this section comes as the result of Experiment Two.

If a transversal intersects two lines in such a way that a pair of corresponding angles have equal measures, then the lines are parallel.

This fact will be used frequently in Chapter 14.

Answers to Questions in Experiment Two

1. Yes. (See criteria outlined in Section 13-9.)
2. Line $\overleftrightarrow{AF}$ should intersect ℓ_1 only in the last case when Angle FAB and Angle b have the same measures.
3. Yes.
4. Yes; the street which is parallel is the one at B which forms equal corresponding angles with the street at A.

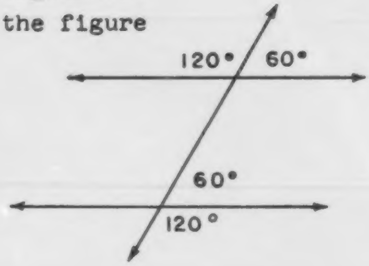
Answers to Exercises 13-10

1. (a),(c),(d),(e). If a transversal intersects two lines in such a way that a pair of corresponding angles have equal measures, then the lines are parallel.
2. (b), (f).
3. a. Yes.

$m\angle 1 = m\angle 2$	Agreed upon
$m\angle 3 = m\angle 2$	Vertical angles have equal measures
$m\angle 1 = m\angle 3$	They are two names for the same number, $m\angle 2$.

   b. 4

4. No; ℓ_1 and ℓ_2 will not intersect. We can show that a pair of corresponding angles are equal in measure, as in the figure at the right.



13-12. Chapter Review.

Answers to Exercises 13-12

1. $m\angle A = 23$; $m\angle B = 90$; $m\angle C = 135$; $m\angle D = 62$
   $m\angle E = 124$.

2. $\angle A$: acute; $\angle B$: right; $\angle C$: obtuse;
   $\angle D$: acute; $\angle E$: obtuse.

3. a. congruent
   b. 180
   c. measure
   d. degree
   e. perpendicular

4. a. $\angle a$, $\angle c$; $\angle b$, $\angle d$; $\angle f$, $\angle h$.
   b. $\angle a$, $\angle b$; $\angle f$, $\angle g$; $\angle e$, $\angle h$.
   c. $\angle a$, $\angle e$; $\angle d$, $\angle h$; $\angle b$, $\angle f$.
   d. $\angle a$, $\angle b$; $\angle b$, $\angle c$; $\angle f$, $\angle g$.
   (There are other possible answers as well.)

5. a. adjacent
   b. vertical
   c. adjacent and supplementary

6. 90°

7. $m\angle b = 110$; $m\angle c = 70$; $m\angle d = 110$; $m\angle e = 70$;
   $m\angle f = 110$; $m\angle g = 70$; $m\angle h = 110$.

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13-13. Cumulative Review.

Answers to Exercises 13-13

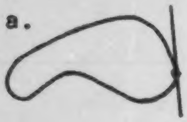
1. {102, 108, 114, 120, 126}

2. 1.8

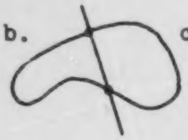
3. a. $\frac{2}{3}$ b. 0 (or $\frac{0}{\text{any counting number}}$) c. $\frac{5}{3}$ d. $\frac{2}{3}$

4. a. 1 sq. ft. b. $\frac{148}{35}$ ft or $4\frac{8}{35}$ ft.

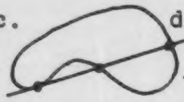
5. a.



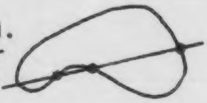
b.



c.



d.



6. a. $\frac{9}{17} > \frac{8}{16}$ b. $\frac{11}{3} = \frac{33}{9}$ c. $\frac{0.3}{0.4} = \frac{0.06}{0.08}$

7. a. 0.8 b. 6.0 c. 0.7 d. 0.1

8. 11 ft.

9. 1 yd ≈ 0.9 meter

$\frac{1}{2}$ mile = 880 yds. The difference is about 770 yards.

1500 meters ≈ 1350 yds.

10. a. C c. the empty set

b. $\overline{BD}$ d. ∠ DCE

e. acute vertical angles

Sample Test Questions for Chapter 13

Teachers should construct their own tests, using carefully selected items from those given here and from items of their own. Careful attention should be given to difficulty of items and time required to complete the test.

(T) 1. Perpendicular lines always meet so that two rays with endpoints at the point of intersection form right angles.

(F) 2. The measure of an obtuse angle is smaller than that of a right angle.

(F) 3. Vertical angles are always supplementary.

(F) 4. Adjacent angles are always supplementary.

(T) 5. All right angles are congruent.

(T) 6. When a line intersects two other lines in two distinct points, it is called a transversal of those lines.

(T) 7. If two lines intersect at a point, the non-adjacent angles are called vertical angles.

(T) 8. If two supplementary angles have the same measure, they are right angles.

(F) 9. The sum of the measures of two supplementary angles is 90.

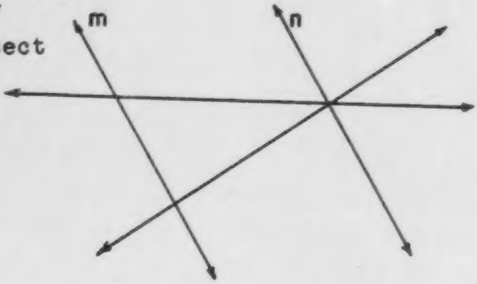
(T) 10. Corresponding angles have interiors on the same side of the transversal.

Multiple Choice

(a) 1. If in the same plane a transversal intersects two lines and the corresponding angles are congruent, then the two lines are . . .

(a) parallel lines.
(b) skew lines.
(c) perpendicular lines.
(d) intersecting lines.
(e) none of the above answers is correct.

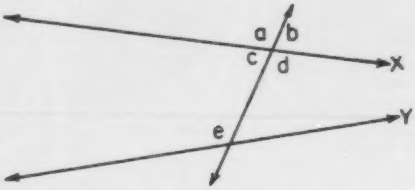
(b) 2. In the figure shown at the right, how many transversals intersect lines m and n?



(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

Questions 3 - 6 refer to the figure below.

(a) 3. Which angle forms a pair of corresponding angles with angle e ?



(a) a
(b) b
(c) c
(d) d
(e) e

(d) 4. Lines x and y are parallel if . . .

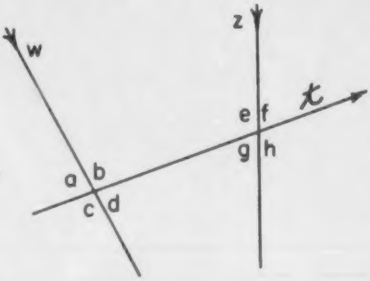
(a) angle a is congruent to angle d.
(b) angle c is congruent to angle e.
(c) angle b is congruent to angle d.
(d) angle e is congruent to angle a.
(e) angle a is congruent to angle b.

(e) 5. Lines x and y are not necessarily parallel and if the measure of angle e is 100, the measure of angle c is . . .

- (a) 100
- (b) 80
- (c) 20
- (d) 10
- (e) unknown

(a) 6. An angle adjacent to angle a is . . .

- (a) b
- (b) d
- (c) e
- (d) angles b, c, and d are all angles adjacent to angle a.
- (e) angle c and angle e.



Using the figure at the right, predict whether lines w and z will be parallel or will intersect. If they intersect, indicate which side of t (above t or below t) the intersection will occur. Fill in the space which correctly completes the following statements. (Do not write on this paper. Indicate answers on answer sheet.)

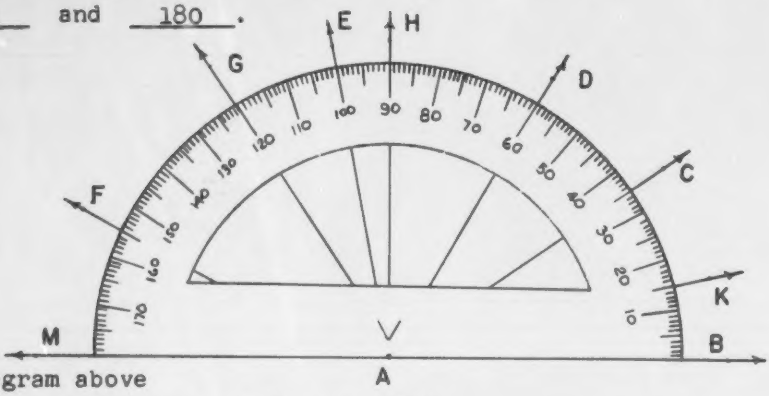
	intersect above t	intersect below t	be parallel
7. If m∠ a = 75 and m∠ e = 75, then the lines will	( )	( )	(X)
8. If m∠ b = 100 and m∠ e = 80, then the lines will	( )	( )	(X)
9. If m∠ c = 120 and m∠ g = 100 then the lines will	( )	(X)	( )

	intersect above t	intersect below t	be parallel
10. If m∠d = 60 and m∠e = 80 then the lines will	( )	(X)	( )

Completion

1. An instrument used to measure an angle is called a (protractor) .

2. An obtuse angle is one whose measure is between 90 and 180 .


E
H
G
D
C
F
K
M
B
A
V
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160
170

In the diagram above

3. What is the measure of ∠FAG? (35)

4. Name a right angle. (∠HAB or ∠MAH)

5. Name an angle which is supplementary to ∠MAC. (∠CAB)

6. Name an angle which is congruent to ∠DAB. (∠MAG or ∠GAD)

7. What is the measure of ∠MAD? (120)

8. What is the measure of ∠HAD? (30)

9. Name an angle which has a measure of 30. (∠HAD or ∠HAG)

10. Name an angle which is supplementary to ∠FAM. (∠FAB)

Chapter 14

POLYGONS AND PRISMS

"Informal geometry" as presented in this chapter, is concerned with the discovery of geometric principles through experimentation and, where feasible, the verification of empirical conclusions by deductive reasoning. Students perform as scientists in collecting data and then perform as mathematicians in the analysis and interpretation of the data they have obtained. Data needed to formulate a statement of a geometric property are obtained by measurement, with protractor or ruler, or by superimposing one figure on another.

Pupils are introduced to the use of deductive reasoning as a method for ascertaining what is true about a geometric figure, arguing from previously stated principles and definitions. We reserve for a later course the systematic organization of geometry as a deductive system, starting with postulates and undefined terms, and developing theorems and definitions on this basis.

The specific purposes for which this chapter was planned are these:

1. To introduce certain geometric concepts and relations as listed below.
2. To give the pupils experience in verification of experimental results by informal deductive argument on the basis of previously stated principles.

The major topics are:

1. The meaning of a converse.
2. The angle and side relationships in a triangle.
3. The angle and side relationships in a parallelogram.
4. Areas of parallelograms and triangles.
5. Definition and volume of a right prism.

Some General Observations and Suggestions.

As in other chapters, precise terminology is emphasized throughout the text material. It is necessary to make this emphasis because many of the words that are casually used by eighth graders are not as clearly understood by their users as we hope. At this level, the consequences of casual language are not always serious but may become so as students proceed in their mathematical studies. All of the terminology developed in previous chapters should be used whenever such usage clarifies and simplifies geometric statements. On the other hand, care should be exercised that in our attempts to be as exact as possible, we do not make a complicated thing out of what may be, essentially, a very simple idea. To avoid this situation, it is suggested that meanings be given first in words of common usage and then in more precise terminology. The translation from common usage to precise usage then becomes an exercise in analytical thinking.

Ideally, once a word that is commonly used is pre-empted for a special meaning in a new vocabulary, the new meaning must be adhered to from that point on. In actual fact, however, it is often difficult to convince an eighth grader that he should do this. In this case we should accept, for the time being, his way of speaking, evaluate and discuss the ideas he is attempting to present, and then encourage him to rephrase his statements according to the more precise language.

Since students often learn best by imitation and habit formation, it is suggested that the teacher become thoroughly familiar with the new terminology and use it at every possible occasion. Through the simultaneous use of both the common and the precise ways of speaking it is hoped the student will become more and more proficient in the latter and grow to appreciate its value until he eventually uses it as a matter of course.

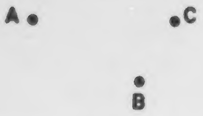
The chapter includes a few deductive developments of a more or less informal nature. One of the problems arising in such a development is that pupils usually fail to appreciate the need for justifying statements with reasons previously adjudged acceptable to the group as a whole. One proposal that might impress them with the fact that only previously stated and accepted properties, definitions, and reasons should be used is to suggest that football and basketball games would be much more interesting if in each game the rules were changed without consulting anybody and that new rules be made up as the game goes along! It might be an interesting game but hardly a fair one! An occasional reminder about "making up rules as you go along" is usually sufficient to make the point desired.

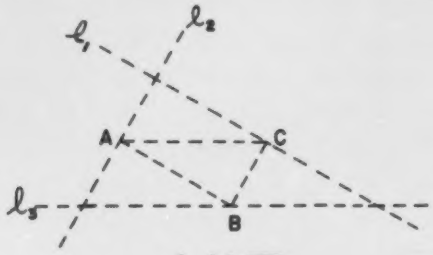
Frequently students are asked to make a general statement about a property or the results obtained through experiment. In the text such statements are partially written so that the grammatical form of the statement is suggested without hinting too strongly at the mathematical ideas involved. Before considering these, students should be encouraged to formulate their own statements of principles and properties but such statements should be very closely examined to ensure that the meaning is precise and clear. When a statement seems satisfactory to all, then show pupils the formulation in the answer section for comparison. They then may use these statements as models in future work in this chapter.

As many as possible of the "Experiments" should be performed in class so that the number of trials will be sufficient to support the conclusions drawn. If it is not possible to do this, those experiments assigned as homework should include an additional instruction to perform a greater number of trials than indicated in the experiment itself. Some experiments require paper cutting. To avoid loss of class time, it is possible in some cases either to assign as homework whatever cutting out must be done or give to each pupil the necessary cut-out already prepared.

About a week prior to the introduction of this chapter, pictures and articles pertaining to plane geometry could be placed on the classroom bulletin board to arouse curiosity and supplement the historical facts briefly mentioned in the first section. Here are two puzzles that will add interest to the display:

1. Drawing a triangle is easy but can you draw a triangle so that each of the dots lettered A, B and C are midpoints of its sides?



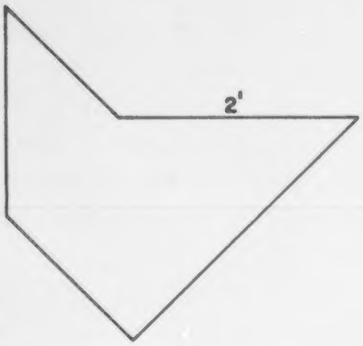


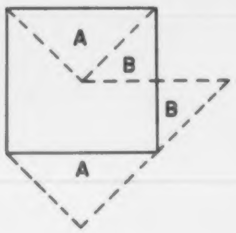
$\ell_1 \parallel \overline{AB}$

$\ell_2 \parallel \overline{BC}$

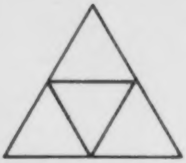
$\ell_3 \parallel \overline{AC}$

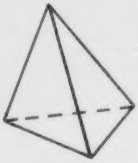
2. A very thrifty cabinet-maker wished to construct a table top two feet square out of a piece of playwood shaped as in the figure. He was able to do this with only two sawings. If you are as clever as the cabinet-maker, you can do the same.





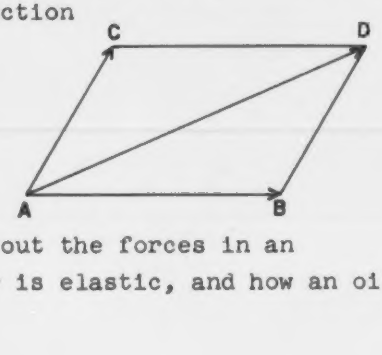
Pictures and geometric diagrams suitable for posting are not always easy to find but some good ones can be obtained from such magazines as "Scientific American", "Fortune", "Popular Science", "Popular Mechanics", "Life", and, of course, some technical magazines. A pattern of "oak tag" paper might be posted along with a completed model for the construction of a regular tetrahedron as an activity in equilateral triangles.





The following hints are given about applications for the purpose of motivation. These may be supplemented by material found in the introductory chapters of plane geometry texts that are available.

There is never an over-supply of good thinkers. The world needs people who can begin with a body of facts, relate them, and think through to logical conclusions. In the aircraft industry there is a great demand for workers trained in geometry because there is a considerable amount of geometric knowledge involved in the construction of an airplane. The main problem is to find out how air will flow about an airplane of given shape moving in a given direction at a given speed. From this the lifting force and the air resistance may be calculated. The parallelogram of forces may be used for an illustration. In order to find the single force equivalent to two forces acting simultaneously at a point we can draw a diagram like this in which the given forces are represented in magnitude and direction by the segments $\overline{AB}$ and $\overline{AC}$. We complete the parallelogram, and the diagonal $\overline{AD}$ gives the magnitude and direction of the resultant force.



Geometry is also used to figure out the forces in an electromagnetic field, and why rubber is elastic, and how an oil company should schedule its production. In the theory of relativity and in the design of agricultural experiments completely different concepts of space are used. Today, the physicist, the chemist, the biologist, the engineer, the economist, the psychologist, and the military strategist use geometry in ways far removed from surveying, some of which were not even discussed or dreamed of only fifteen years ago.

14-1. Kinds of Triangles.

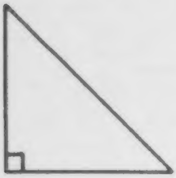
Concepts to be developed:

1. There are three sets of triangles determined according to the measures of their sides.
    - (a) The set of isosceles triangles has as members triangles which have two sides that are equal in length.
    - (b) The set of scalene triangles includes triangles which have no two sides with the same measure.
    - (c) The set of equilateral triangles includes triangles which have three sides equal in length.
2. An angle and a side of a triangle are said to be opposite each other if their intersection contains just the endpoints of the segment referred to as the side.
3. If two sides of a triangle are equal in length, then the angles opposite these sides have equal measures.
4. An equilateral triangle is a special kind of isosceles triangle.

Students should try to answer questions in the text as they reach them and not read ahead for the answers. Answers are included in the text so that, if desired, all or a portion of the text may be assigned for reading outside of class.

Have a number of soda straws measured and creased before class begins. This exercise may seem easy to most pupils but even the brighter ones will jump to incorrect conclusions about a straw divided into three pieces of 2", 3", and 5". Soda-straw figures give the class an opportunity to handle triangles in regions of space other than the chalkboard or the drawing paper. Make certain that the soda-straw figures are understood to be merely representations of triangles and not actual triangles. This particular exercise should lead students to deduce that the sum of two sides of a triangle must be greater than the third.

Answers to Exercises 14-1a

1. Isosceles
2. 
3. Scalene. No two sides have the same length.
4. a. Isosceles.
   b. Equilateral. All three sides would have the same length.
5. a. Equilateral.
   b. Isosceles. At least two sides have the same length.
6. a. Isosceles
   b. Scalene
   c. Equilateral
   d. Isosceles
   e. Equilateral
   f. No triangle is formed
   g. No triangle is formed

Answers to Questions in Experiment One

1. $\overline{AB}$ and $\overline{AC}$
2. $\angle C$
3. $\angle B$
4. They are equal.

This seems to be true: If two sides of a triangle are equal in length, then the angles opposite these sides are equal in measure.

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Answers to Exercises 14-1b

1. a. $m \angle A = m \angle C$
   b. $m \angle R = m \angle Q$
   c. $m \angle X = m \angle Z$

2. $\angle A$; $\overline{HQ}$; $\angle X$; $\overline{YZ}$

3. Yes; every equilateral triangle has at least two equal sides.
   No; an isosceles triangle need only have two equal sides.

14-2. Converse of a Statement.

Concepts to be developed:

1. The converse of an "if -- then" sentence is formed by reversing the order of the "if-clause" and the "then-clause".

2. The converse of a statement may be true or false regardless of whether the original statement is true or false.

Suggestions:

It might prove helpful to clarify the meaning of "true" or "false" before taking up the exercises. Without going into details of logic, it should be sufficient to propose that a "false" statement is not considered true if at least one counter-example can be found.

For example, examine the statement: "The set of whole numbers is closed under subtraction". As a counter-example one might say, "there is no whole number which added to five gives three and, therefore, (3 - 5) is not a name for a whole number". This one counter-example is all that is needed to deny the quotation. Emphasize that only one counter-example is needed to prove a statement false.

While one counter-example can be used to show that a statement is false, it is a great deal more difficult to show that a statement is true. In this section we do not expect that the students or teacher will prove that statements are true. At the end of Section 14-7 a geometric proof is given for the theorem: An exterior angle of a triangle is equal in measure to the sum of the measures of the two non-adjacent interior angles of the triangle.

It is helpful to use many non-mathematical examples to develop the concept of a converse. It is also important to emphasize the fact that the converse of any statement, true or false, may also be true or false. For example, consider the statement;

If the sun shines,
then I go for a walk.

Assume this statement is true. The converse is:

If I go for a walk,
then the sun shines.

This converse is not necessarily true in that one might walk even when it rains!

Have the students compose their own statements and converses, and examine these to test whether each is true or false.

In forming converses we sometimes run into difficulty for grammatical reasons. Thus there is no problem in composing the following pair of statements:

STATEMENT	CONVERSE
If $\angle a$ and $\angle b$ are right angles, then $\angle a$ and $\angle b$ have the same measure.	If $\angle a$ and $\angle b$ have the same measure, then $\angle a$ and $\angle b$ are right angles.

Now let us write the original statement in a slightly different form:

STATEMENT: If two angles are right angles, then they have the same measure.

If the converse of this statement is written in accord with the procedure outlined in this chapter, we have the following:

CONVERSE 1: If they have the same measure, then two angles are right angles.

In this form, the word "they" may cause difficulty in the interpretation of the sentence. The word "they" refers to the right angles, but the converse would be in better grammatical form if written as follows:

CONVERSE 2: If two angles have the same measure, then they are right angles.

The pronoun "they" caused no difficulty when it appeared in the original statement. There the clause containing the antecedent came before the clause containing the pronoun. In converse 1 the clause containing the pronoun came first and it is not clear what "they" refers to.

In this section we do not call the student's attention to this source of difficulty but expect him to clarify the converses he formulates by making suitable grammatical adjustments.

The difficulty occurs in a slightly different way in such statements as:

STATEMENT: If two lines are parallel, then the lines do not have a point in common.

CONVERSE 1: If the lines do not have a point in common, then two lines are parallel.

Here no pronoun occurs, but the expression "the lines" which occurs in the second clause of the statement has for its antecedent the expression "two lines" which occurs in the first clause. The difficulty is solved by writing the converse as:

CONVERSE 2: If two lines do not have a point in common, then the lines are parallel.

The problem can be completely avoided by substituting for the statement the following:

> If lines ℓ and m are parallel, then lines ℓ and m do not have a point in common.

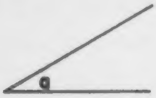
The converse can then be formed by strict adherence to the rule given in the text as:

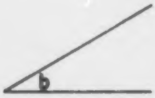
> If lines ℓ and m do not have a point in common, then lines ℓ and m are parallel.

Answers to Exercises 14-2a

1. If $\angle a$ and $\angle b$ have the same measure, then $\angle a$ and $\angle b$ are right angles.

   $m\angle a = m\angle b$, but they are not right angles.





2. a. False
   b. True
   c. True
   d. True
   e. True (if no amputees)
   f. True

3. a. If Blackie is a cocker spaniel, then Blackie is a dog. (true)
   b. If we cannot see the sun, then it is night. (false)
   c. If it is a holiday in the United States, then it is July 4th. (false)
   d. If Robert is the tallest boy in his class, then he is the tallest boy in his school. (false)
   e. If an animal has four legs, then the animal is a horse. (false)
   f. If an animal has thick fur, then the animal is a bear. (false)

4. a. True
   b. True
   c. False
   d. True
   e. False
   f. True

5. a. If a figure is a simple closed curve, then the figure is a circle. (false)
   b. If a figure is a triangle, then the figure is a simple closed curve composed of three line segments. (true)
   c. If two angles are right angles, then the angles have equal measures. (true)
   d. If two lines do not have a point in common, then the lines are parallel. (In a plane--true; In space--false)
   e. If two angles are adjacent, then they are supplementary. (false)

f. If two angles are supplementary, then they are adjacent right angles. (false)

6. If two angles of a triangle are equal in measure, then the sides opposite these angles are equal in length. (true)

Answers to Questions in Experiment Two

1. Angles of the same measure: $m\angle B = m\angle C$.
2. $\overline{CF}$; $\overline{BF}$.
3. They would have to be equal in measure.

This seems to be true: If two angles of a triangle are equal in measure, then the sides opposite these angles are equal in length.

Answers to Exercises 14-2b

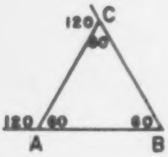
(Class Discussion)

1. If two parallel lines are intersected by a transveral then a pair of corresponding angles formed are equal in measure.
2. The corresponding angles should have the same measure.
3. True
4. If two non-parallel lines are intersected by a transversal then a pair of corresponding angles formed are not equal in measure. (Note: Accept answers to Problems 1 and 4 as given in the students' own words. Do not insist upon precise language here - but insist that the statement be correct.)

Answers to Exercises 14-2c

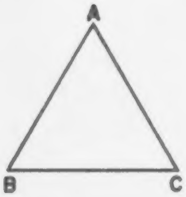
1. a. HR = MR b. AC = BC c. ST = SQ
2. Isosceles
3. a. If two parallel lines are cut by a transversal then a pair of corresponding angles formed are equal in measure.
   b. If two sides of a triangle are equal in length, then the angles opposite these sides are equal in measure.
   c. $m\angle 1 = m\angle 3$.

4. Equilateral. The figure at the right shows that the measure of each angle is 60.



5.

1.	$m\angle A = m\angle B$	1.	Agreed upon.
2.	$m\ \overline{BC} = m\ \overline{AC}$	2.	Experiment 2.
3.	$m\angle C = m\angle B$	3.	Agreed upon.
4.	$m\ \overline{AB} = m\ \overline{AC}$	4.	Experiment 2.
5.	$m\ \overline{BC} = m\ \overline{AC}$ $= m\ \overline{AB}$	5.	All names for the same number $m\ \overline{AC}$.



6. Scalene: No two angles of the same measure.
Isosceles: At least two angles of the same measure.
Equilateral: All three angles have the same measure.

14-3. Angles of a Triangle.

The major concept to be developed in this section, by Experiment Three, is that the sum of the measures of the angles of a triangle is 180. To save time each student might be asked to come to class with a triangle ready to be cut. After the sum of the measures of the three angles is found by each member of the class, the results can be listed on the board. These should be close to 180.

The experiments described in Problems 9 and 10 of Ex. 14-3 may also be carried out in class. Wax paper is an especially good medium for completing the paper folding exercise described in Problem 10. (For additional information on paper folding see the booklet prepared by the National Council of Teachers of Mathematics, Paper Folding, by D. Johnson.)

Answers to Exercises 14-3

1. a. $m\angle x = 92$ c. $m\angle x = 103$
   b. $m\angle x = 29$ d. $m\angle x = 16$

2. a. $m\angle x = 40$, $m\angle y = 40$ d. $m\angle x = 30$, $m\angle y = 48$
   b. $m\angle x = 45$ e. $m\angle x = 150$, $m\angle y = 75$
   c. $m\angle x = 48$, $m\angle y = 132$

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3. 60

4. a. 65 and 65 or 50 and 80
   b. Yes

5. a. $m\angle x = 45$
   $m\angle y = 100$
   $m\angle z = 35$

   b. $m\angle x = 28$
   $m\angle y = 130$
   $m\angle z = 22$

   c. $m\angle x = 63$
   $m\angle y = 63$
   $m\angle z = 77$

   d. $m\angle x = 27$
   $m\angle y = 35$
   $m\angle z = 118$

6. a. $m\angle x = 37$
   $m\angle y = 37$
   $m\angle z = 53$

   b. $m\angle x = 30$
   $m\angle y = 30$
   $m\angle z = 60$

7. $m\angle a = 101$
   $m\angle b = 39$
   $m\angle c = 51$

8. $m\angle a = 120$
   $m\angle b = 60$
   $m\angle c = 60$
   $m\angle d = 60$
   $m\angle e = 120$
   $m\angle f = 60$
   $m\angle g = 60$
   $m\angle h = 120$
   $m\angle i = 60$
   $m\angle j = 120$
   $m\angle k = 60$
   $m\angle l = 60$
   $m\angle m = 60$
   $m\angle n = 120$
   $m\angle o = 60$
   $m\angle p = 120$

9. $m\angle 1 + m\angle 2 + m\angle BAC = 180$.

10. When folded the three angles of the triangle meet at point G and appear to have a total measure of 180.

14-4. Polygons.

This is the place to review some of the terminology studied last year in Chapter 7 with respect to simple closed curves. The names of different polygons are introduced, although many of these may already be familiar to the student. Some members of the class may enjoy hunting in the literature for the names of polygons with more than six sides;

7 sides - heptagon
8 sides - octagon
9 sides - nonagon
10 sides - decagon

Answers to Exercises 14-4

1. c.

2. c.

3. a and c are polygons; b is not
   Not the union of segments.

4. a. hexagon c. pentagon
   b. hexagon d. quadrilateral

5. ABEF, BCED, ACDF.

6. a. $\overline{AD}$, $\overline{BC}$
   b. $\overline{AB}$, $\overline{DC}$
   c. $\overline{AD}$, $\overline{BC}$
   d. $\overline{AB}$, $\overline{DC}$

14-5. Parallelograms.

Important ideas in this section:

1. A parallelogram is a quadrilateral whose opposite sides are parallel.
2. The diagonal of a parallelogram divides the interior into two congruent triangles.
3. The opposite angles of a parallelogram are equal in measure.
4. The opposite angles of a parallelogram are equal in measure.

Most of the work in this section needs to be done in class in the form of laboratory experiments, although some time can be saved by having drawings made and cut out at home.

In particular, the two methods for constructing parallelograms need to be carefully developed if skill in construction is to be expected.

Answers to Exercises 14-5a
(Class Discussion)

1. a, d
2. ABFE, BCDE, ACDF
3. AJGF, AJID, AJHE, DIHE, DIGF, EHGF, BMEK, BKIL, ILME.

Answers to Exercises 14-5b

1. Yes, because each is one of two equal parts.
2. The area of each one is equal to one-half the area of the rectangle. (Students may also say that the two pieces are congruent.)
4. A diagonal of a parallelogram divides the interior of the parallelogram into two triangular pieces each of which has an area equal to one-half the area of the parallelogram.

Answers to Questions in Experiment Four

The lengths of the segments are:

Number of the figure	Length of opposite sides:	
	First side	Second side
1	1" $1\frac{1}{4}$"	1" $1\frac{1}{4}$"
2	2" 1"	2" 1"
3	2" 1"	2" 1"
4	$3\frac{1}{2}$" $\frac{1}{2}$"	$3\frac{1}{2}$" $\frac{1}{2}$"
5	1" 1"	1" 1"

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1. They are equal.
2. The opposite sides must be equal in length.

Answers to Questions in Method I

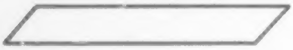
1. Corresponding angles. Yes.
2. Yes. When a transversal intersects two lines so that a pair of corresponding angles have equal measures, then the lines are parallel.
3. Yes. Same reason as for 2.
4. Yes. The opposite sides are parallel.

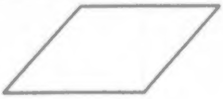
Answers to Questions in Method II

1. Yes. The figure was constructed in this manner.
2. A parallelogram.
3. Parallelogram.
4. This statement can not be shown to be true by any number of examples; it must be proved in general.

Answers to Exercises 14-5c

1.

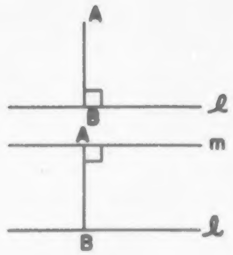




2. Yes. Because of the results of Experiment Four.
3. Determined. All points of the angle rays are not points of the parallelogram.
4. The measures of opposite angles are equal.

14-6. Distance to a Line.

At times we refer informally to a segment as being a distance. However it is the length of a segment which is the distance. Thus the distance from A to ℓ is not the segment $\overline{AB}$, but the length of this segment. Similarly, the distance between the parallel lines ℓ and m is the length of segment $\overline{AB}$.



Answers to Questions in 14-6

$\overline{EF}$ does not make right angles with ℓ_1 and ℓ_2.

$\overline{JK}$ does not have an endpoint on each of the parallel lines, thus J is not on ℓ_1.

Answers to Exercises 14-6

1. a. 1" b. $1\frac{1}{2}$" c. $\frac{3}{4}$"
2. a. $1\frac{1}{4}$" b. 2" c. $1\frac{1}{2}$"
3. a. $1\frac{1}{2}$" b. $\frac{3}{4}$"
4. a. 1" b. $\frac{3}{4}$" c. $\frac{1}{2}$"

14-7. Proof.

The concept of proof is a difficult one for youngsters to understand. They see no reason to prove statements which appear obvious. The suggested use of optical illusions helps to show that we should not make judgments on the basis of looks alone. Interesting bulletin board displays can be made on this topic with the class asked to find additional optical illusions.

Some members of the class may enjoy doing some outside reading on prime numbers in connection with this section.

With a very slow class, the sample proof given may prove too subtle and need not be stressed. It is used as a sample only, and is not needed for the continuity of the chapter.

14-8. Area of a Parallelogram.

Experiment Five makes a dynamic presentation of the method for finding the area of a parallelogram. It is helpful for the teacher to have a large model prepared for demonstration purposes; possibly for use on a flannel board if one is available.

Answers to Exercises 14-8a

1. a. A, B, C, D
   b. $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$
   c. $\overline{AB}$ and $\overline{CD}$, or $\overline{AD}$ and $\overline{BC}$
   d. No

2. a. $\overline{AD}$ or $\overline{BC}$
   b. $\overline{XY}$ or $\overline{HZ}$
   c. $\overline{RS}$ or $\overline{MQ}$
   d. Any side.

3. a. ABCD, HMRS, UVTQ.
   b. Yes. The distance between a set of parallel lines remains unchanged.

4. a. ABCE, ABFD.
   b. $\overline{AE}$, $\overline{BC}$.

Answers to Questions in Experiment Five

1. A rectangle.

2. Yes; the two parts were merely interchanged.

3. Yes.

4. Yes.

5. No; we have arrived at our conclusion by considering an arbitrary figure. The same process can be used for any parallelogram.
   This seems to be true:
   The number of square units of area in a parallelogram is the product of the number of linear units in the base and the number of linear units in the altitude to this base.

Answers to Exercises 14-8b

1. a. 15 sq. yds.
   b. 12 sq. ft.
   c. 18 sq. ft.
   d. 36 sq. ft.
   e. 50 sq. in.

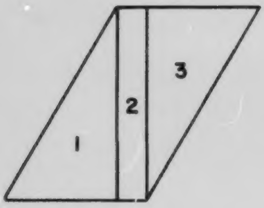
2. $7\frac{1}{2}$ sq. in.

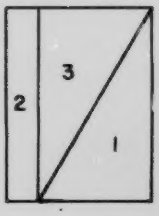
3. $(4)(3) + (4)(2) + (4)(1) = 24$ sq. in., or
   $(4)(6) = 24$ sq. in.

4. $(3)(2\frac{1}{2}) + (2)(2\frac{1}{2}) + (1)(2\frac{1}{2}) = 15$ sq. in., or
   $(6)(2\frac{1}{2}) = 15$ sq. in.

5. All have the same altitude and base. The area is 20 sq. in. for each.

6.





14-9. Areas of Triangles.

The procedure for finding the area of a triangle depends upon dividing a parallelogram into two equal parts. Thus it might prove wise to review briefly the results of Exercises 14-5b at this time.

The material of this section can easily be developed as a laboratory experiment, in class or at home. The conclusion reached is:

The number of square units of area in a triangle is one-half the product of the number of linear units in the base and the number of linear units in the altitude to this base.

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Answers to Exercises 14-9

1. a. 10 sq. in.
   b. 32 sq. cm.
   c. 28 sq. yd.
   d. 136 sq. in.
   e. 65 sq. ft.
2. 30 sq. yd.
3. No. The openings have a total area of 4 sq. ft., which is 8 sq. ft. less than is required.
4. Area of window = 60 sq. ft.
   60 sq. ft. at $50 per sq. ft. = (60)(50) = 3000
   Cost of window = $3000.
5. 3750 sq. ft., 11,250 sq. ft.
6. Area = 2 sq. in.

14-10. Right Prisms.

Before starting this section it might be wise to review the material developed in Chapter 12, especially with respect to terminology.

Section 14-10 has been written from an intuitive point of view. In particular, no explicit discussion has been given of the concepts of lines perpendicular to planes or of perpendicular planes, although both are involved in the idea of a right prism.

Students should be encouraged to construct models as indicated by the plane patterns given in this section.

Answers to Exercises 14-10a

1. Figure 1: a. triangular right prism
   b. A, B, C, D, E, F.
   c. ABFE, BCDF, ACDE.
   d. ABC, EFD.

   Figure 2: a. pentagonal right prism
   b. A, B, C, D, E, A', B', C', D', E'.
   c. ABB'A', BCC'B', CC'D'D, DD'E'E, EE'A'A.
   d. ABCDE, A'B'C'D'E'.

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Figure 3: a. triangular right prism
b. A, B, C, X, Y, Z.
c. ABYX, BCZY, CZXA.
d. ABC, XYZ

Figure 4: a. rectangular prism
b. A, B, M, H, C, F, R, D.
c. ABCD, BMFC, MFRH, HRDA, ABMH, DCFR.
d. ABMH, DCFR, or any other pair of opposite sides.

Figure 5: a. triangular right prism
b. A, X, Y, V, H, T.
c. AXHV, XYTH, YTVA.
d. AXY, VHT.

Answers to Exercises 14-10b

1. a. 3 sq. in.
   b. 8 sq. in
   c. 5 in.
   d. 40 cubic inches
2. a. 32 cubic inches
   b. 18 cubic inches
   c. $37\frac{1}{2}$ cubic inches
3. a. 30 cubic inches
   b. 56 cubic inches
   c. 126 cubic inches
4. a. 216 cubic inches
   b. 18 fish
5. a. No. Volume of tank is 40 cubic feet, ten cubic feet short of requirements.
6. Triangular right prism.
7. Pentagonal right prism.

page 160-163: 14-12,14-13

14-12. Chapter Review.

Answers to Exercises 14-12

1. a. equilateral b. scalene c. isosceles d. isosceles

2. a. If a figure is a simple closed curve then the figure is a triangle. (false)
   b. If $\angle a$ and $\angle b$ are corresponding angles, then $\angle a$ and $\angle b$ are equal in measure. (false)

3. a. 30 b. 90 c. 40

4. a. $\overline{AC}$ b. $\overline{AC}$ c. $\overline{AB}$

5. a. 15 sq. in. b. 20 sq. in. c. $17\frac{1}{2}$ sq. in.

6. a. $7\frac{1}{2}$ sq. in. b. 7 sq. in. c. 10 sq. in.

7. a. 80 cu. in.
   b. 36 cu. in.

8. 21 sq. in. It is not possible to determine the shape of the base from the information given.

14-13. Cumulative Review.

Answers to Exercises 14-13

1. $\{1, 7, 49\}$

2. One of the factors is zero.

3. 640

4. a. T
   b. T

5. a. $0.3\overline{8}\ldots$
   b. $0.0\overline{714285}\ldots$

6. a. obtuse, 135 d. acute, 70
   b. right, 90 e. obtuse, 110
   c. acute, 45

7. a. Separates space into two half-spaces.
   b. Separates $\overleftrightarrow{AD}$ into two half-lines.
   c. Separates plane ABC into two half-planes.

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8. As lines are drawn through R (as shown) they will cut both lines ST and VW in sets of points which can be matched one-to-one.

9. V = 336 cu. in.

10. 90

11. none of them

12. a. .00027
    b. 17.56

13. {0, 21, 42, 63, 84}

14. $105.30

15. $71\frac{13}{24}$ sq. ft.

16. 15 sq. in.

17. The area of the triangle = $\frac{1}{2}$ the area of the parallelogram.

Sample Test Questions for Chapter 14

This set of questions should not be used as a chapter test. Teachers should construct their own tests from those given here and from their own. There are too many questions here for one test.

True-False.

F 1. If one angle of an isosceles triangle has a measurement of 66°, one of the other two angles must have a measurement of 66°.

T 2. A statement may be true while its converse is false.

T 3. Pairs of corresponding angles with equal measures are formed when a transversal intersects two parallel lines.

T 4. A statement and its converse may both be true.

T 5. If a triangle has two sides with equal measures, then it has two angles with equal measures.

T 6. The sum of the measures of the three angles of a triangle is equal to 180.

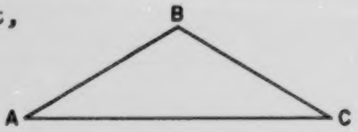
F 7. An equilateral triangle is also a scalene triangle.

F 8. The converse of a false statement is always false.

F 9. If a triangle has only two sides with equal measures, it can have three angles with equal measures.

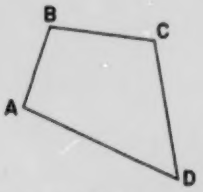
T 10. An equilateral triangle is also an isosceles triangle.

T 11. In the figure at the right, A, B, and C are names for the vertices of the triangle.



T 12. All parallelograms are quadrilaterals.

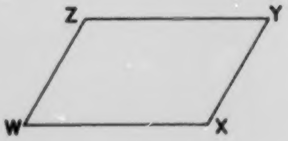
F 13. In the four sided figure at the right, if m($\overline{AB}$) = m($\overline{CD}$) and if $\overline{AD}$ and $\overline{BC}$ are parallel, then the figure is a parallelogram.



T 14. A triangle may never have two angles whose measures are 90 .

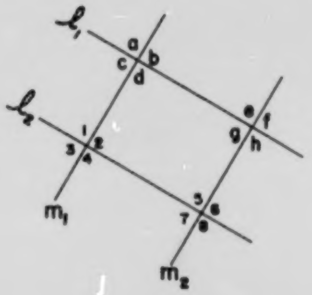
F 15. If the measures of the four sides of a parallelogram are equal, then the figure is always a square.

F 16. In the figure at the right, if $\overline{WX}$ and $\overline{YZ}$ are parallel, then WXYZ is a parallelogram.



F 17. It is possible to draw a triangle whose sides measure 4 inches, 2 inches, and 1 inch.

The figure shown at the right consists of two pairs of parallel lines. Use this for items 18-22.



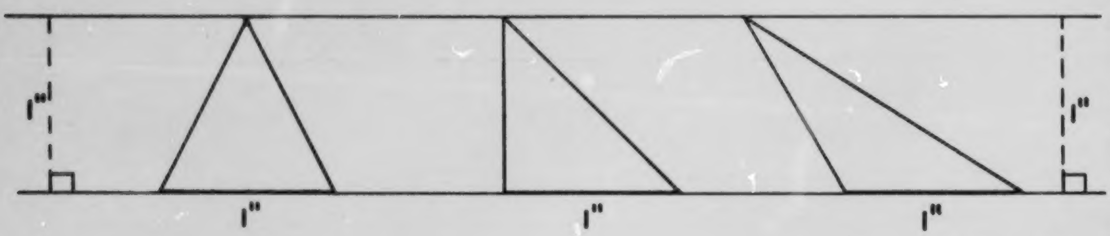
T 18. $m\angle a = m\angle 8$

F 19. $m\angle 3 = m\angle h$

T 20. The figure contains more than 16 pairs of equal angles.

T 21. If $m\angle 7 = m\angle 8$ then all the measures of the angles shown in the figure are equal.

T 22. The triangles shown below all have the same area.



Multiple Choice.

1. If the measure of one angle of a scalene triangle is 50, which of the following statements is always true?
   a. One of the other angles has a measure of 90.
   b. One of the other angles has a measure of 50.
   (c) c. The sum of the measures of the other two angles is 130.
   d. Two of the sides are equal.
   e. One of the other angles has a measure of 130.

2. If the measure of one angle of a triangle is equal to the measure of another angle in the triangle, then:
   a. the measures of the three sides of the triangle are equal.
   b. no two measures of sides are equal.
   (d) c. the measures of the sides opposite the angles whose measures are equal are not equal.
   d. the measures of two sides are equal.
   e. none of the above statements is correct.

3. If two sides of a triangle have lengths of three inches and four inches, the third side could measure ...

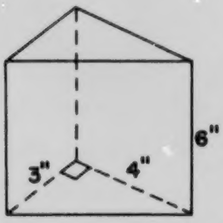
- a. one inch.
- b. seven inches.
- (e) c. less than one inch.
- d. more than seven inches.
- e. none of the above answers is correct.

4. Which of the following statements is not correct?

- a. A polygon is a simple closed curve.
- b. A polygon is the union of line segments.
- c. Every simple closed curve is a polygon.
- (c) d. A pentagon is a polygon.
- e. Every polygon has at least three sides.

5. The volume of the right prism in the figure is:

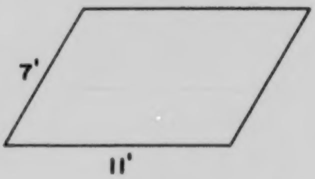
- a. 13 cu. in.
- b. 36 cu. in.
- (b) c. 42 cu. in.
- d. 72 cu. in.
- e. none of these is correct.



6. Which of the following requirements is not necessary for a prism to be a right prism?

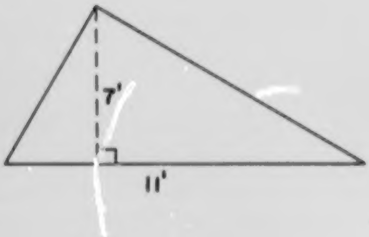
- a. Its bases must lie in parallel planes.
- b. Its bases must be alike in size and shape.
- (d) c. All faces, except the bases, must be rectangles.
- d. The bases must be rectangles.
- e. The name of a right prism depends on the shape of the base.

7. The area of the parallelogram shown at the right may be found by ...



- a. adding 11 and 7.
- b. multiplying 11 and 7.
- (e) c. multiplying 11 and 7 and dividing the product by two.
- d. multiplying (11 + 7) by $\frac{1}{2}$.
- e. none of the above answers is correct.

8. The area of the triangle shown at the right may be found by ...



I. $\frac{11 \times 7}{2}$

II. $\frac{1}{2} \times 11 \times 7$

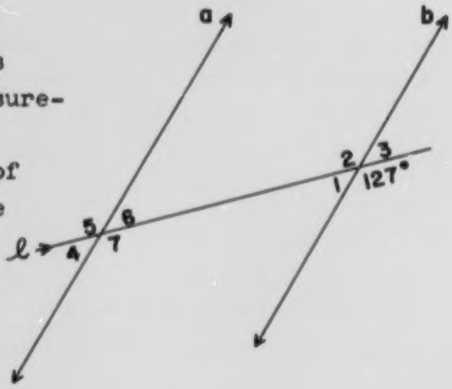
III. $\frac{1}{2}(11 \times 7)$

IV. $7 \times \frac{1}{2} \times 11$

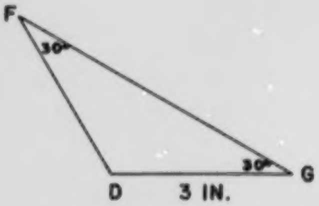
a. only I and III are correct.
b. only II and IV are correct.
(e) c. only I and II are correct.
d. only I, II, and III are correct.
e. all of the above are correct.

Completion.

In the figure at the right lines a and b are parallel and ℓ is a transversal. What are the measurements of each of the following angles? Note: the measurement of one of the angles is given in the figure.



1. angle 6 (53°)
2. angle 2 (127°)
3. angle 1 (53°)
4. angle 5 (127°)

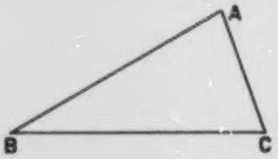


5. In the figure at the right the length of segment $\overline{DF}$ is (3 in.).

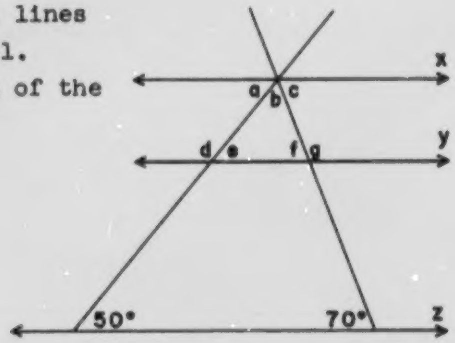
6. The measure of one of a pair of vertical angles is 40; the measure of the other one is (40).

7. Two lines intersect at point A. If the measure of one angle formed is 70, an adjacent angle has a measure of (110).

8. In the triangle at the right, m∠ABC = 30, m∠BCA = 70. What is the measure of angle CAB? (80)



In the figure at the right, lines x, y, and z are parallel. What is the measure of each of the following angles?



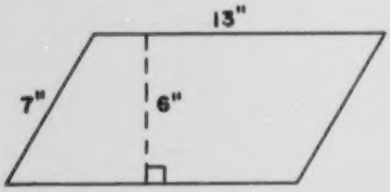
9. angle a (50)
10. angle b (60)
11. angle c (70)
12. angle d (130)
13. angle e (50)

14. The converse of the statement "If m∠a = m∠b, then ∠a and ∠b are vertical angles" is: (If ∠a and ∠b are vertical angles, then m∠a = m∠b.)

15. The converse of the statement "If you like mathematics then you are intelligent" is: (If you are intelligent, then you like mathematics.)

Find the areas of the following:

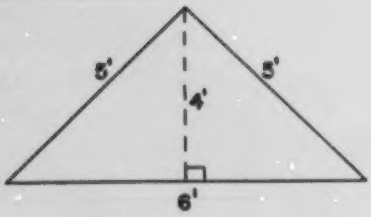
16.



PARALLELOGRAM

Area = (78 sq. in.)

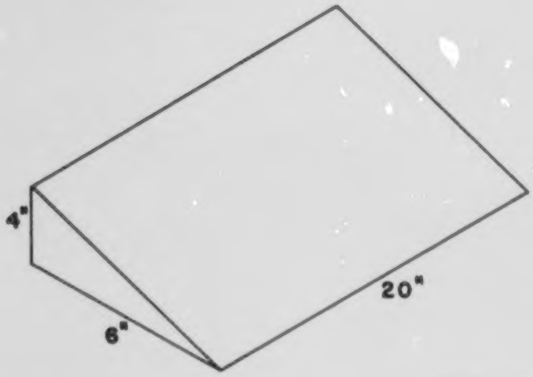
17.



Area = (12 sq. ft.)

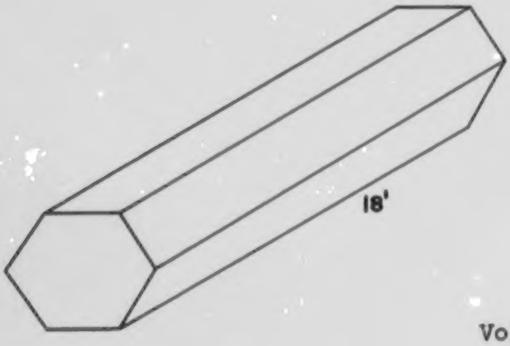
Find the volumes of the following:

18.



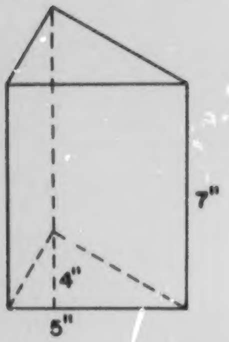
Volume = (240 cu. in.)

19. Area of the base is $2\frac{1}{2}$ square feet.



Volume = (45 cu. ft.)

20.



Volume = (70 cu. in.)

Chapter 15

CIRCLES

The main purposes of this chapter are:

1. To acquaint the pupils with the circle, its length and area, and with some of its elementary applications to cylindrical solids.
2. To develop precision of expression and thought, and to develop geometric awareness and intuition including understanding of appropriate methods of study.

Except for the study of the circle itself, the general purposes of this chapter are like those of Chapters 4 and 7 in Volume I and Chapters 13 and 14 in Volume II. The teacher may choose to reread the introduction to the commentaries for these chapters.

An effort has been made to use accurate statements concerning the distinction between length and measure of a length. The number r is not the radius but rather the measure of the radius.

Since in ordinary speech distinctions have not been made between length and measure of a length, many teachers may find themselves interchanging these terms. An important distinction is that we multiply numbers but not lengths. For many purposes this idea is not too important for junior high school pupils. Teachers are advised to note such subtleties of thought and to make some effort to use precise language. They should not be alarmed at their own failure to do so consistently. They should not insist on more than reasonable precision from students.

15-1. Circles and the Compass.

You may wish to begin this chapter by calling attention to prevalence of the circle in industrial and in decorative design. Some pupils enjoy bringing in pictures illustrating the use of the circle. In the history of civilization, discovery of the use of the wheel is ranked in importance with discovery of a method for producing fire.

The problem in the text is planned to provide practice in use of the compass and to help the student formulate a correct definition of a circle. It is important that each pupil have a compass and learn to use it correctly and with some dexterity.

The common errors of beginners in use of a compass are:

1. not using a hard sharp pencil
2. drawing the circle first in one direction and then in the other
3. pressing the point of the compass too hard into the paper
4. altering the distance between the point and the pencil tip while drawing the circle
5. pausing while drawing the circle
6. not grasping the compass at the top.

Answers to questions in text of 15-1

All the points that are 3 inches from P lie on a circle. There are infinitely many points of this circle, so the method of locating them with a ruler is not possible. Stress that a circle is a set of points, just the points on the rim or edge of a circular disk and not all the points on the disk and its interior.

$\overline{PS}$ is 3 inches in length. Remember that when you speak of the measure of line segments you are speaking of a number and you write $PS = 3$. A circle has infinitely many radii, since it is composed of infinitely many points and each one can be joined by a line segment to the center. Each radius of a circle has the same length.

Answers to Exercises 15-1a

In following the directions for drawing figures, pupils should learn to label the figure at each stage, as subsequent

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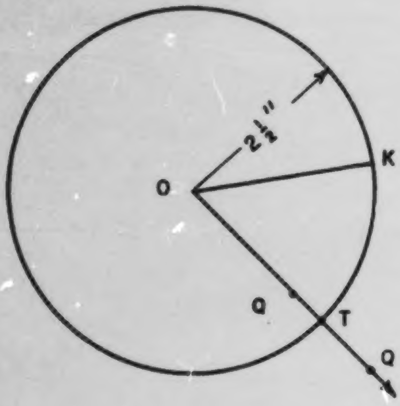
directions are given in terms of the labels specified.

1. a.–b.



c. The length of $\overline{OK}$ is $2\frac{1}{2}$ inches.

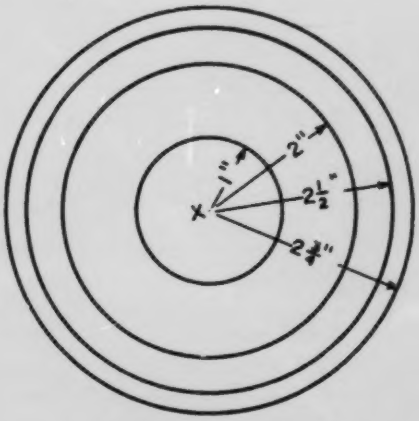
d. $2\frac{1}{2}$ inches

e. Note Q may be either inside or outside the circle.

g. T is on circle.

h. $\overline{OT}$ is a radius.

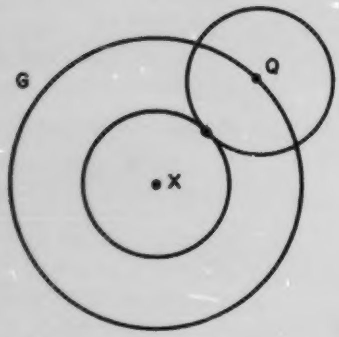
2.



3. The purpose of this exercise is to practice using the compass. Let the children discover where the centers of the circles are by trial and error. The final results are not important.

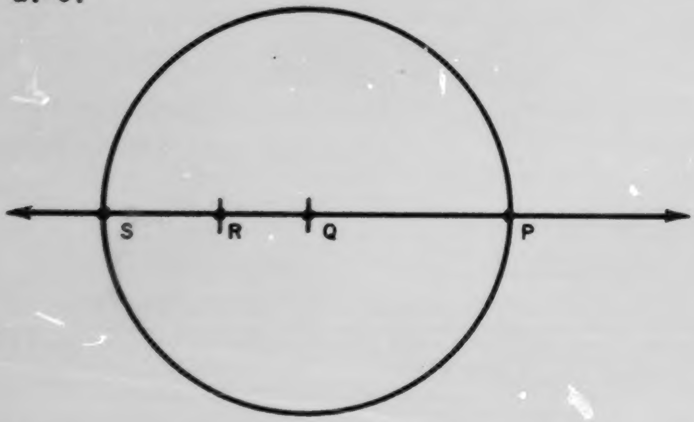
Answers to Exercises 15-1b

1. a.-b.-c.



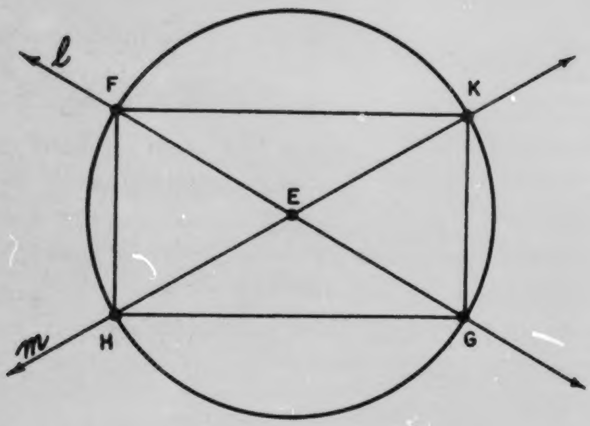
d. The intersection seems to be a single point.

2. a.-e.



f. The points are S and P.

3. a.–f.

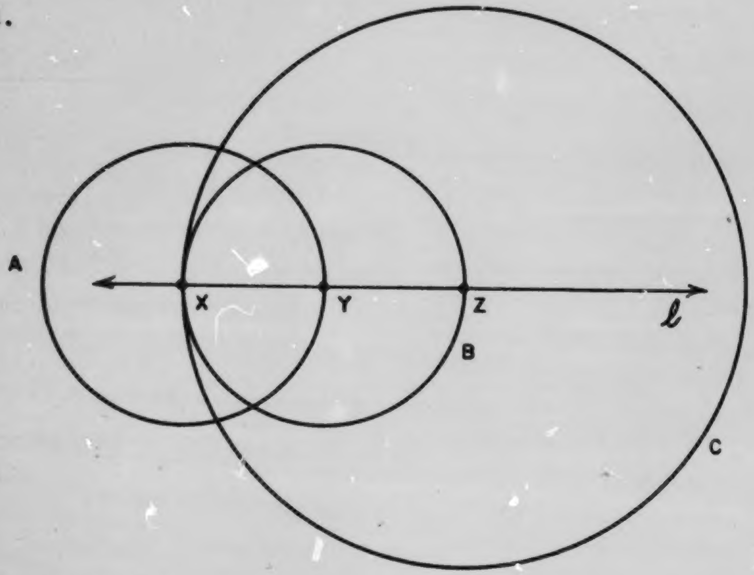


d. Two points

e. Two points

g. HFKG seems to be a rectangle. (HFKG is a rectangle. If the diagonals of a quadrilateral are the same length, and also bisect each other, then the quadrilateral is a rectangle.)

*4. a.–d.



e. Intersection set is two points.

f. Point X

15-2. Interiors and Intersections.

This section is designed to help the pupils to discover what is meant by the interior of a circle and the exterior of a circle. They should be able to decide when a point is in the interior of a circle and when a point is in the exterior of a circle. There is an opportunity to review the notion of intersection and union of sets. Pupils should be repeatedly encouraged to translate the written idea into a diagram in a step-by-step manner as they read. This cannot be overemphasized, for it is applied throughout geometry.

Answers to questions in text.

Points C and Q are on the circle.
Points P, A, and B are in the interior of the circle.
Points D and E are in the exterior of the circle.

Answers to Exercises 15-2a

1. interior of circle P.
2. exterior of circle R.
3. the interior of the circle.
4. less than
5. equal to
6. greater than

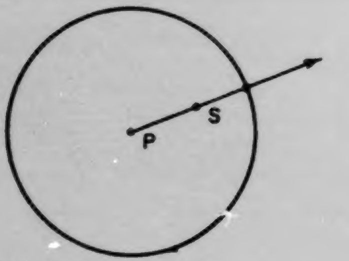
It may be necessary to review more of the work on intersections and unions. Sufficient material can be found in Chapters 4 and 7 in Volume I on Non-Metric Geometry.

Answers to Exercises 15-2b

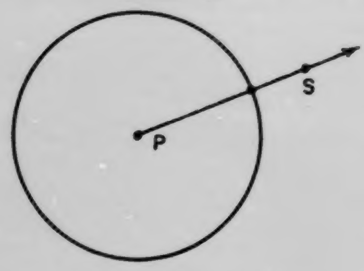
1. a. Yes. The single point Q lies on the circle and also on $\overrightarrow{PQ}$.
   b. (Circle P) $\cap$ $\overrightarrow{PQ}$ = Point Q
2. (Circle P) $\cap$ $\overrightarrow{QP}$ = Point Q and one other point. (The second point needs a label.)
   b. Two.

3. Point S can be chosen either in the interior (case 1) or on the circle or in its exterior (case 2).

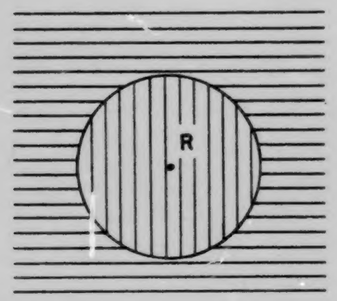
Case 1



Case 2

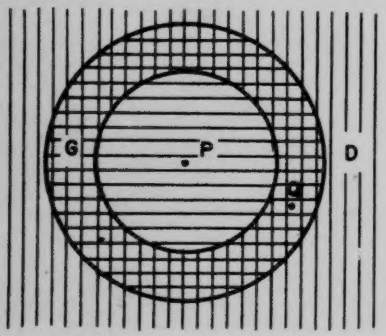


a. one
b. two
c. no.
d. In case 1. no points
In case 2. one point



4. a. No points
b. The empty set.
c. No points.
d. The empty set.
e. The area which is shaded vertically and its boundary. The best way to describe this set of points accurately is to say "the union of circle R and its interior."

*5.



e. The intersection is shaded with both horizontal and vertical lines.

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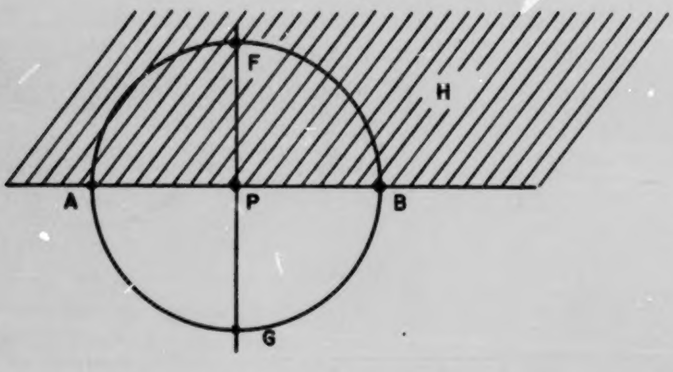
This exercise could be done on the blackboard with colored chalk to show the intersections.

Answers to Exercises 15-2c

1.

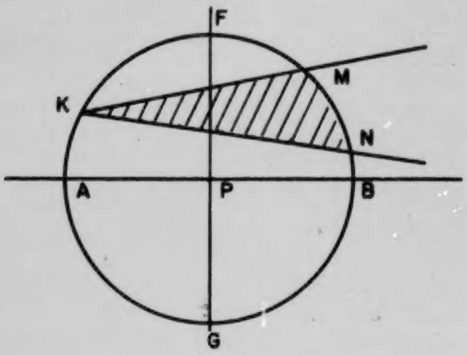


2. a. Circle $P \cap H =$ the set of all points on arc $\overset{\frown}{AFB}$ not including the points A and B. (The students have not met this notation but they should be able to tell which points belong to the intersection.)

   b. The points A, G, and P do not belong to the intersection. The point F does belong to the intersection.

   c. You cannot count all the points of the intersection.

3. a.—b.



   c. Intersection of the circle and the angle MKN consists of the three points M, N, and K.

d. The intersection of the two interiors has been shaded. No points on the boundary belong to the intersection.

*4. a. Same as interior of $\angle$ BPF.

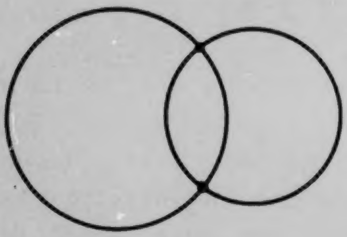
b. The quarter-circle in the upper right, excluding the endpoints B and F.

c. Four of the quarters may be identified. Besides the one in (b), we have the intersection of the circle and the interior of the angle $\angle$ APF, in the upper left; the intersection of the circle and the interior of $\angle$ APG; the intersection of the circle and the interior of $\angle$ BPG. In each case, a quarter can also be described as the intersection of the circle and two half-planes; for example, the lower left quarter is the intersection of the simple closed curve, AFBG, the A-side of $\overleftrightarrow{FG}$, and the G-side of $\overleftrightarrow{AB}$.

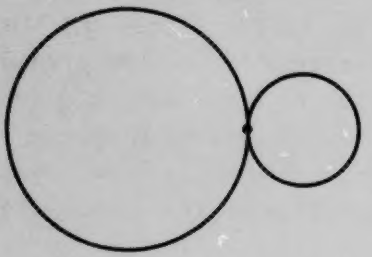
d. Two portions of the circle might be called halves. The upper half, for example, is the intersection of the circle and the half-plane H. In the next section, we learn that this intersection together with its endpoints A and B, is a semicircle. Another is the lower half, the intersection of the circle and the G-side of $\overleftrightarrow{AB}$. Another is the intersection of the circle and J. The fourth one shown is the intersection of the circle and the A-side of $\overleftrightarrow{FG}$. In each of the four cases, the curve together with its endpoints is a semicircle. The pupils may wish to include the endpoints when they describe a "half" of the circle. This problem helps blaze the trail for certain notions in Section 4 of this chapter.

Answers to Exercises 15-2d

1.



2.



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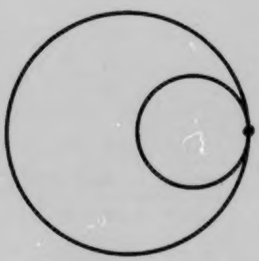
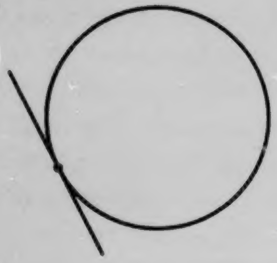
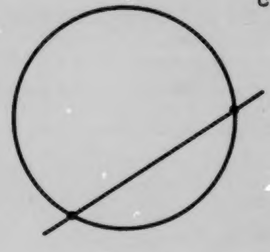


3. a.

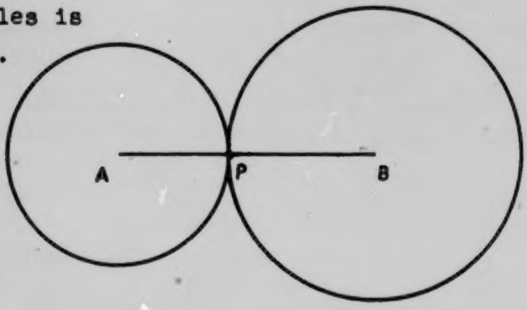


b.

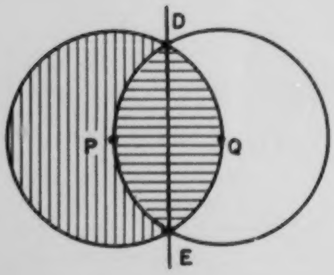


c. Impossible. It is not possible for more than two points to be in the intersection set.

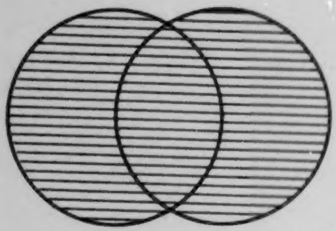
4. The intersection of the two circles is the point P.



5. a. Consists of two points. (They are labeled D and E here.)
   b. No; two points determine just one line.
   c. Horizontally shaded in the figure.
   d. Vertically shaded in the figure.

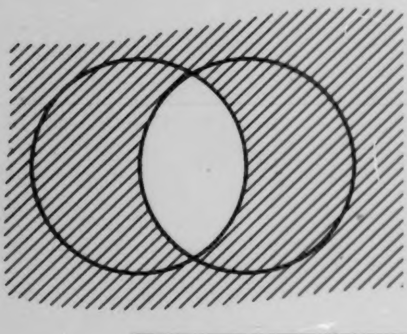


   e.



6. Note that the definition of concentric circles is in this Exercise.
   a. Empty set.
   b. Intersection of the interior of the outer circle and the exterior of the inner circle.
   c. Same as the exterior of the outer circle

7.



15-3. Diameters.

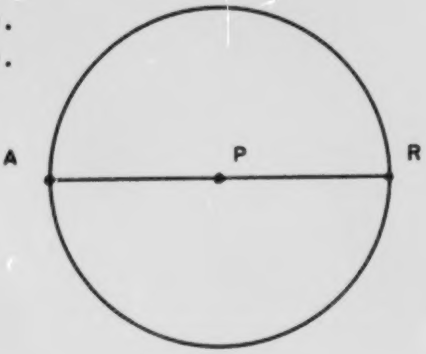
A circle can have infinitely many diameters but they will all have the same length.

The six radii shown in the figure are $\overline{PA}$, $\overline{PW}$, $\overline{PN}$, $\overline{PB}$, $\overline{PV}$, and $\overline{PM}$.

The length of the diameter is twice the length of the radius of the same circle. Pupils should learn the relationship $d = 2r$, $r = \frac{1}{2}d$.

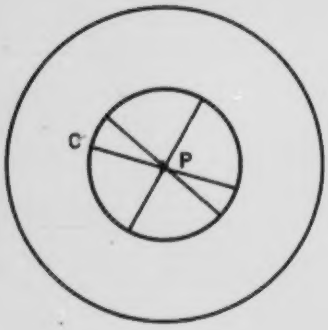
Answers to Exercises 15-3

1. The diameter is 6 inches.
2. The radius is 2 inches.
3. a. $d = 6$ d. $r = 1\frac{1}{2}$
   b. $r = 5$
   c. $d = 9$
4. a. one, $\overline{PR}$ in our figure.
   b. one, $\overline{AR}$ in our figure.



5. 9

6.



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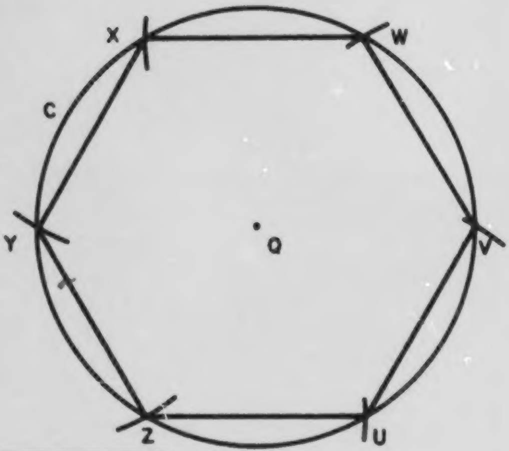
7. All the line segments $\overline{UV}$, $\overline{VW}$, $\overline{WX}$, etc. have the same length as the radius of the circle. Be sure to emphasize that the arc lengths are longer than a radius. Many children think this proves that the circumference of the circle is 6r.



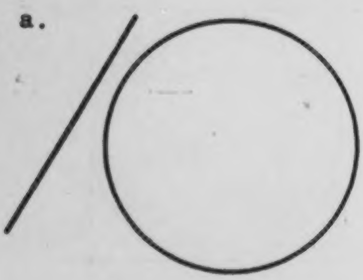
15-4. Tangents.

The next set of exercises should enable the pupils to understand the three possibilities for the intersection of a line and a circle. Namely,

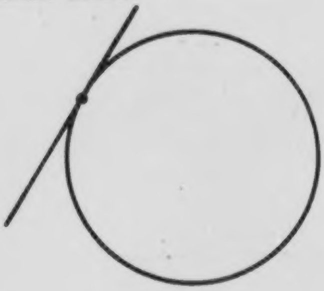
1. The intersection may be the empty set. In this case there is no point on the circle that is also on the line.
2. The intersection may be a set containing a single point. In this case we say the line is tangent to the circle. We call this point, the point of tangency.
3. The intersection may be a set containing two points. This case may be the one the pupil usually thinks of when he talks about a line intersecting a circle.

Answers to Exercises 15-4

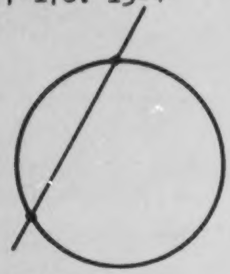
1. a.



b.



c.



d. Impossible. A circle and a line can have no more than two points of intersection.

2. Part b.

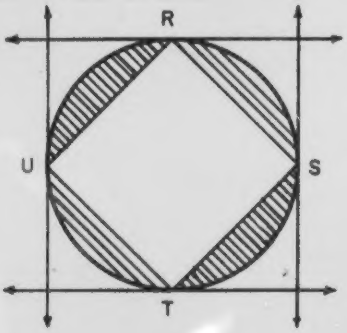
3. a. 4 b. 1 c. $\overleftrightarrow{AR}$, E is point of tangency.
$\overleftrightarrow{AT}$, F is point of tangency.
$\overleftrightarrow{RT}$, S is point of tangency.

4. a. 3 b. 4 c. 5

It is correct to refer to these circles as being inscribed in the polygons.

5. a. {R, S, T, U}
   b. Point T
   c. $\overleftrightarrow{HE}$, $\overleftrightarrow{EF}$, $\overleftrightarrow{FG}$, $\overleftrightarrow{HG}$
   d. R, S, T, U.

6. a. None
   b. The empty set
   c. Intuitively it is composed of the four corner regions. An attempt on the pupil's part to give a careful description is the one quoted in the problem: "the intersection of the exterior of the circle and the interior of the square EFGH." A purpose of this problem is to point out the advantages of our vocabulary in enabling us to say just what we mean with the greatest possible ease.
   d. As in the preceding part, the best word description is given in the problem itself. The intersection is shaded in the figure.



7. They are perpendicular.

*8. Each diameter is the union of two different radii on one line. If the intersection of two segments on one line consists of a common endpoint, then the length of their union is the sum of the lengths of the two segments. Since all radii have the same length, the measure of a diameter is twice the measure of a radius in the same circle. Since this applies to all diameters in the same circle, all of them have the same length.

15-5. Arcs.

This section extends the idea of separation. The pupil learned that only one point was necessary to separate a line into two half-lines. Here the pupil should see that one point is not sufficient to separate a circle into two parts.

Definitions and qualifying descriptions which are especially important in this section are arc, symbolism for arcs.

Answers to questions in text

Q does not separate the circle into two parts. We need two points. The points on $\overset{\frown}{MEN}$ are all points on the part of the circle between M and N which includes point E, and the endpoints M and N.

$\overset{\frown}{NEM}$ contains the same points as $\overset{\frown}{MEN}$.
$\overset{\frown}{MAN}$ is the same as $\overset{\frown}{NAM}$.
A and E are the endpoints.

Answers to Exercises 15-5

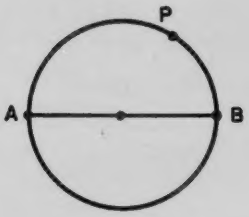
1. a. $\overset{\frown}{AMC}$ b. $\overset{\frown}{NCM}$ c. $\overset{\frown}{ANC}$, $\overset{\frown}{ACM}$, $\overset{\frown}{AMC}$, $\overset{\frown}{AMN}$, etc.
   d. $\overset{\frown}{NAM}$, $\overset{\frown}{NMC}$, $\overset{\frown}{CNM}$, $\overset{\frown}{CAM}$, etc. ...

2. a. A, B, and C ; endpoints are A and C.
   b. A, B, C, D, and E; endpoints are A and E.
   c. A, B, C, D, and E; endpoints are A and E.
   d. F, A, B, C, D, and E; endpoints are F and E.
   *e. There are two possible arcs, both with endpoints B and E. One contains A, F, B, and E. The other contains C, D, B, and E.

   Be sure to emphasize that all these arcs contain countless points only a few of which are labeled.

3. a. none
   b. A, B
   c. none
   d. none

4. a. two
   b. They divide the circle into two parts of equal length.



   d. Yes
   e. No
   f. {A, B}

15-6. Central Angles.

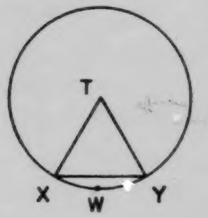
The definition of central angle is especially important if arcs are measured by arc-degrees. The topic of arc-degrees has not been treated here. (See SMSG *Mathematics For Junior High School*, Volume 1 part II, pp. 486-88 for a brief treatment.) The exercises are intended to prepare pupils for the next section on circumference.

Answers to Exercises 15-6

1. a. $\overline{AP}$, $\overline{BP}$, $\overline{CP}$, or $\overline{DP}$
   b. $\overline{AB}$, $\overline{CD}$
   c. $\overset{\frown}{ADB}$, $\overset{\frown}{DBC}$, $\overset{\frown}{ACB}$, $\overset{\frown}{CAD}$
   d. $\angle APD$, $\angle DPB$, $\angle BPC$, $\angle APC$

2. a. $\overset{\frown}{CDE}$
   b. $\overset{\frown}{BDE}$
   c. the empty set
   d. Point C
   e. $\overset{\frown}{ACE}$
   f. $\overset{\frown}{ACD}$
   g. $\overset{\frown}{BDF}$
   *h. $\overset{\frown}{ABC}$ and $\overset{\frown}{DEF}$. To say $\overset{\frown}{ABC} \cup \overset{\frown}{DEF}$ is the best way to describe this set of points.

3. a. 1 inch
   b. Equilateral
   c. $\angle YTX$
   d. 60
   f. $\overset{\frown}{XWY}$ is longer than $\overline{XY}$



15-7. Length of a Circle.

The relation between the length of a circle and the length of its diameter is developed by means of two experiments. The first method consists of measuring the circumferences and diameters of circles, examining the measures for a possible relation, and observing that, in any circle, the ratio $\frac{c}{d}$ is approximately the same number. It is good procedure to have the pupils report their measurements, tabulate them, and then propose relations which might exist. This is to be preferred to guiding them too soon and too firmly toward the ratio.

The second method is important as a basis for underlining the notion that π is a number, and that the symbol "π" is to be regarded as a numeral. This is the first number of the set of irrational numbers the pupils have met. They should develop the notions that (1) such numbers correspond to points on the number line, and (2) that a decimal expression for the number, to any desired precision, may be written.

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Although π is not a rational number, the assumption is tacitly made here that it behaves like an ordinary number in combining with other numbers; that is, the commutative, associative, and distributive properties for addition and multiplication hold. Many pupils are interested in the fact that mathematicians have studied the properties of this number for centuries, and continue to do so; and are entertained by seeing the decimal for π to many decimal places. A decimal expression for π to 10,000 decimal places was published in the year 1958. This is mentioned in an article by F. Genuys in Chiffres 1 (1958), 17–22.

Pupils may be interested in seeing certain decimal expressions for π. Here is the decimal for π to fifty-five places. 3.14159 26535 89793 23846 26433 83279 50288 41971 69399 36510 58209... (The three dots at the end indicate that the decimal expression continues indefinitely.)

Pupils may be amused my a mnemonic device for remembering the first figures in the decimal for π. It is a rhyme in which the number of letters in each word indicates the digit:

"See, I have a rhyme assisting
My feeble brain, its tasks sometime resisting."

Pages 24 and 25 of The Wonderful World of Mathematics by Lancelot Hogben should provide interesting reading for pupils.

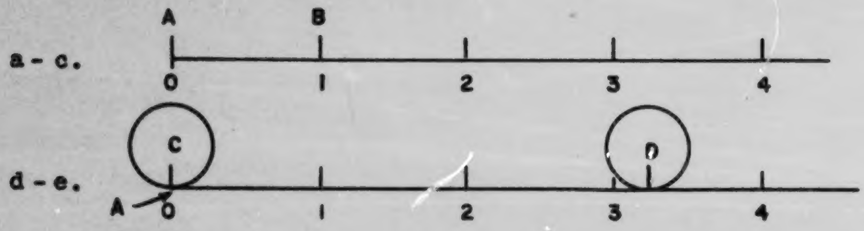
Experiment 1

a.–d. Answers will depend on measurements obtained.

f. Differences are not the same.

g. Ratios will vary but should be distributed around 3.1 or 3.2. If a correctly computed ratio is larger than 3.4 or smaller than 2.9, suggest that the object be measured again.
The results should suggest that the circumference of each circle is about 3 times as long as its diameter.

Experiment 2



f. Between 3 and 4. Estimates will probably vary from 3.0 to 3.3.

g. $\overline{AD}$

h. Length of the circle is about 3.0 - 3.3 times the length of the diameter.

Answers to Exercises 15-7a

1. 3.143
2. 3.142
3. The difference is .001
4. 3.14
5. 3.14
6. Yes
7. Yes
8. 21.98 or 22 feet
9. 94.2 or 94 feet
10. 3.82 or 3.8 feet
11. a. $\pi = \frac{c}{d}$ b. $c = \pi d$ c. $\frac{c}{\pi} = d$

*12. a. $\frac{c}{r} = 2\pi$ b. $r = \frac{c}{2\pi}$ c. $\frac{c}{2r} = \pi$

Answers to Exercises 15-7b

1.

	circle	r	d	C
a.	A	5	10	31.4
b.	B	4.2	8.4	26.376 or 26
c.	C	3.98 or 4.0	7.96 or 8.0	25
d.	D	6.25 or 6	12.5	39.250 or 39

2. a. 44 inches
   b. 132 feet
   c. 42 inches
   d. 7 feet
   e. 66 inches
3. 110 inches
4. 37.71 inches, or 38 inches
5. 690.80 feet, or 691 feet
6. 19.72 inches, or 20 inches
7. 94.2 feet, or 94 feet
8. a. 62.0 inches
   b. 7.75 inches or 8 inches
9. 12.56 inches, or 13 inches
10. a. Length of circle C $\approx$ 44 inches
    Length of circle D $\approx$ 31.4 inches or 31 inches
    b. The linear measure of arc $\overset{\frown}{QYR}$ is 8.8 inches or 9 inches.
    The linear measure of arc $\overset{\frown}{SXT}$ is 6.2 inches or 6 inches.

15-8. Area of a Circle.

Chapter 12 explains the basic method for finding the area of a closed region. In the case of a circle we prefer to have a method for computing the area in terms of the radius. Only one approach to the development of the relationship $A = \pi r^2$ is presented in this section. This is not an application of the basic method of measurement such as is given in Mathematics for Junior High School, Volume 1, part 2, pp. 500, but is a very plausible argument. It should be done as a class exercise. The segments of the circle could be cut out of construction paper and pasted on the blackboard.

Answers to Exercises 15-8a

1. a. $\overline{AB}$ is a diameter.
   b. $\overline{AB}$ divides the region into 2 parts of equal area.

2. a. 10 but this number is completely arbitrary.
   The greater the number the better the approximation.
   *b. 18°
   c. radii of the circle
   d. $\frac{1}{2}$ the circumference, therefore πr

4. a. Yes
   b. parallelogram.
   c. $\frac{1}{2}$ the circumference or πr
   d. $\overline{BP}$ is a radius, so $BP = r$

5. a. base times altitude.
   b. $\pi r \cdot r$ or πr^2

6. a. $AB = \pi r$
   b. $BP = r$
   c. $\pi r \times r$
   d. πr^2

7. The area of circle is π times the area of square with side r.

Answers to Exercises 15-8b

1. 154 square feet
2. 113 square feet
3.

r	d	c	A
20	40	124	1240
4	8	24.8	49.6
0.015	0.030	0.093	0.0006975
12	24	74.4	446.4

4. The circular skillet has the greater frying surface since $50.24 > 49$. Area of circular skillet ≈ 50.24 sq. in.
Area of square skillet ≈ 49 sq. in.
Note that this has the reverse answer to the example in the text. Here the circular skillet has the larger area.

5. Approximately 113 square inches.

6. a. Approximately 28.26 square inches.
   b. Approximately 113.04 square inches.
   c. The area of a circle that has a radius of 6 inches is about 4 times as large as the area of a circle that has a radius of 3 inches.
   d. The area is multiplied by 4.
   e. The area is multiplied by 4.

7. Approximately 154 square feet.
8. Approximately 39 square feet.
9. Approximately 1231 square inches.
10. a. Approximately 154 square inches
    b. Yes. Area of square $\approx$ 196 square feet.
    $\frac{3}{4}$ of the area of square $\approx$ 147 square feet
    which is just a little less than 154 square feet.

*11. $40 \cdot 30 - 7^2 \cdot \pi$, to the nearest square foot, 1046.

*12. a. 16π
     b. 4
     c. 8
     d. 8π
     e. 4π
     f. $8 + 4\pi$

*13. 3

15-9. Volume of a Cylindrical Solid.

The pupil is already familiar with standard methods of finding the volume of a prism. This section will extend the principle learned for prisms to a solid which has two circular bases. The pupil should realize that no new process is involved. The pupil should learn the principle that the measure of the volume is the measure of the area of the base times the measure of the altitude.

Exercises 15-9a develop the essential properties of a cylinder. They are summarized in the definition which follows the exercise.

Answers to Exercises 15-9a

1. 2
2. Circular region
3. Yes.
4. The planes are parallel.
5. The base lies directly above the other.
6. a. Yes
   b. Yes
   c. $\{A, C\}$

Answers to Exercises 15-9b

1. a. $\pi \cdot 2^2 \approx 12.56$ square inches
   b. 12.56 cubic inches
   c. Yes, because the altitude is 1 inch,
2. Approximately 25.12 cubic inches
3. Approximately 37.68 cubic inches
4. a. twice
   b. 3
5. 226.08 cubic feet
6. 62.80 cubic feet

Sometimes it is a good idea to use π as a numeral instead of a decimal approximation. We say that the answer is expressed in terms of π.

At times you may encourage your students to delay the use of a decimal approximation for π until the last step of the computation.

Answers to Exercises 15-9c

1. $V \approx 3391.2$ or Volume is 3391.2 cubic feet.
   (3391, 3390, or 3400 could be used.)
2. $V \approx 198$ or Volume is 198 cubic inches.
3.

r	d	h	V
5	10	12	300π
1	2	20	20π
$4\frac{1}{2}$	9	$5\frac{1}{2}$	$\frac{891}{8}\pi$
$1\frac{1}{4}$	$2\frac{1}{2}$	16	25π

4. The can that is 3 inches high and 4 inches in diameter holds more than the other can. Its volume is 37.68 cubic inches. The volume of the other can is 28.26 cubic inches. $37.68 > 28.26$.
5. a. Volume is 6 cubic feet.
   b. 47 gallons (or 45 gallons if one computes from 6 cubic feet.)
6. a. $V \approx 90\pi$
   b. $V \approx 180\pi$
   c. $V \approx 360\pi$
   d. $V \approx 720\pi$
   *e. 1. multiplied by two (or doubled)
       2. multiplied by four
       3. multiplied by eight
7. 56,520
8. $V \approx 3768$ or Volume is about 3770 cubic inches.

15-10. Surface Area of a Cylindrical Solid.

Experiment 3

b. rectangular region.
c. height or altitude
d. length or circumference
e. π
f. $\pi \cdot d \cdot h$
g. $2 \cdot \pi \cdot r \cdot h$
h. Yes

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Answers to Exercises 15-10

1. a. 1256 square feet
   b. 1884 square feet
2. a. 75.36 square centimeters
   b. 89.49 square centimeters
3. 377 square inches

*4. 3.5 gallons

5. a. πr^2
   b. $2\pi r^2$
   c. $2\pi rh$
   d. $2\pi r^2 + 2\pi rh$

15-12. Chapter Review.

Answers to Exercises 15-12

1. a. A circle is a simple closed curve.
   b. Any line segment through the center of a circle and having its endpoints on the circle is a diameter of the circle.
   c. The diameter of a circle is twice as long as the radius of the same circle.
   d. A part of a circle is called an arc.
   e. The length of a circle is called its circumference.
   f. A circle whose diameter is 5 inches has a radius that is $2\frac{1}{2}$ inches.
   g. A line that intersects a circle in exactly one point is called a tangent.

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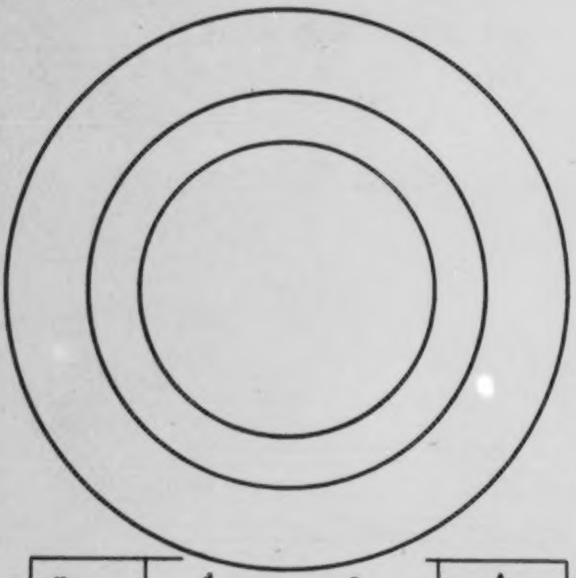
2.



3.

r	d	c	A
14	28	86.8	607.6
7	14	43.4	151.9
8.2	16.4	51	208.44
17.5	35	108.5	949.38
10	20	62	314

4. Circumference is about 154 inches.
5. Diameter is 24 feet.
6. $C \approx 47.10$ (or circumference is about 47 inches.)
7. $4\frac{3}{4}$ feet.
8. 3.5 inches
9. a. Area is 1256 square feet.
    b. Area is 314 square feet.
    c. The area of the circle with radius 20 feet is four times the area of the circle with radius 10 feet.
10. The area of the table top is about 572 square inches.
11. Area is about 346.19 square inches.
12. Area is about 12.56 square feet.
13. Volume is about 62.8 cubic inches.

*14. Approximately 19.

15-13. Cumulative Review.

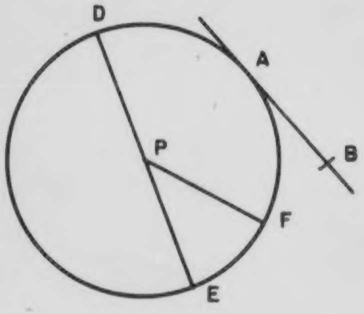
Answers to Exercises 15-13

1. (b) is false.
2. {1, 7, 11, 77}
3. 3 + 23, 7 + 19, 13 + 13
4. 12
5. 1
6. a. 0.048
   b. 628.38
7. a. 1653.2
   b. 0.3491
8. $4\frac{1}{2}$%
9. \$1.25
10. a. $\angle ABF$ d. the empty set
    b. $\overline{BC}$ e. Point E
    c. Point D f. Points D and E
11. The region that is doubly shaded is the intersection of the exterior of C_2 and the interior of C_1.
12. a. 720 cm. c. $20\frac{1}{2}$ feet
    b. 4.7 cm. d. 22440 feet
13. b
14. There are 16 of each kind in part a, b, and c, but 8 in part d.
15. The length is about 207 inches.

Sample Test Questions for Chapter 15

Note: Teachers should construct their own tests, using carefully selected items from those given here and from items of their own. Careful attention should be given to difficulty of items and time required to complete the tests.

1. In the figure at the right, find an illustration for each term in the column at the left, and write its name on the line after the term.



a.	Diameter	$\overline{DE}$
b.	Radius	$\overline{PF}$ or $\overline{PE}$ or $\overline{PD}$
c.	Arc	$\overset{\frown}{EF}$ or $\overset{\frown}{AF}$ or $\overset{\frown}{DA}$ or $\overset{\frown}{DF}$, etc.
d.	Tangent	$\overleftrightarrow{AB}$
e.	Semicircle	$\overset{\frown}{DAE}$ or $\overset{\frown}{DE}$
f.	Obtuse angle	$\angle DPF$
g.	Two line segments of equal length	$\overline{DP}$, $\overline{PF}$, $\overline{PE}$
h.	Central angle	$\angle DPF$, $\angle FPE$
i.	Two points on the circle which separate A and E	D and F
j.	Acute Angle	$\angle FPE$

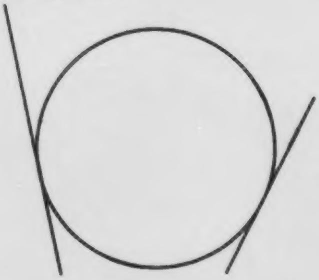
2. Ellen's belt is 21 inches long and just fits around her waist. If her waist were a perfect circle, what would its diameter be?	Answer: 6.7 inches
3. A value for π used by the ancient Babylonians is $\frac{355}{113}$. How different is this from the value we use, to two decimal places?	Answer: $\frac{355}{113} = 3.1415929...$ $\frac{22}{7} = 3.1428571...$ $\pi = 3.1415926...$
*4. A merry-go-round has three rows of horses. The outer row is 6 feet farther out than the inner row. If you sit on an outside horse, how much farther do you ride in one turn of the merry-go-round than if you sit on the inside horse?	Answer: 12π feet or 37.68 feet.

5. The diameter of a certain circle is 60 centimeters. What is the area of this circle? (Use the decimal approximation of π to the nearest hundredth.)

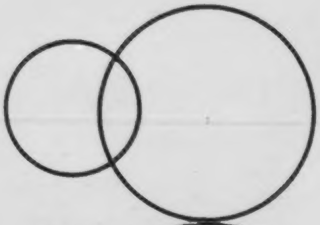
Answer: 2826 square centimeters.

6. Draw a circle and two segments such that each of the segments is tangent to the circle.

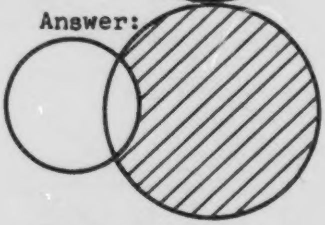
One possible answer:



7. Shade the intersection of the interior of the larger circle and the exterior of the smaller circle.

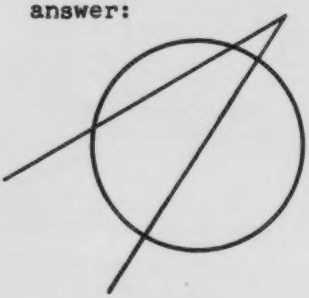


Answer:

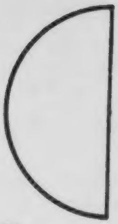


8. Draw a circle and an angle such that their intersection consists of four points.

One possible answer:

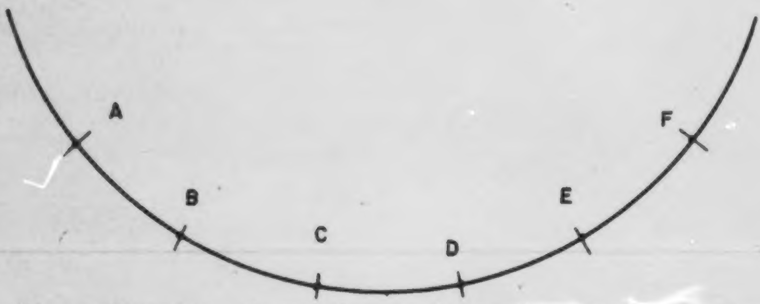


*9. The simple closed curve in the figure consists of a semicircle and a diameter. The radius of the circle is r units of length. Find the area of the interior of the simple closed curve. (The answer should be expressed in terms of r.)



Answer:
$\frac{1}{2}\pi r^2$ units of area.

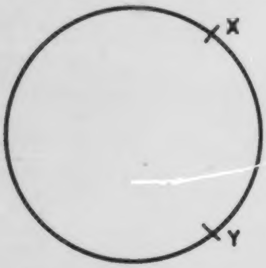
10.



Given the arc shown above, determine the following:

		Answer:
a.	$\overset{\frown}{BE} \cap \overset{\frown}{CF}$	a. $\overset{\frown}{CDE}$
b.	$\overset{\frown}{AF} \cap \overset{\frown}{DE}$	b. $\overset{\frown}{DE}$
c.	$\overset{\frown}{AC} \cap \overset{\frown}{BF}$	c. $\overset{\frown}{BC}$
d.	$\overset{\frown}{CE} \cap \overset{\frown}{DE}$	d. $\overset{\frown}{DE}$

11. With the aid of a sketch, define "arc".



Answer:
The points X and Y determine an arc on the circle. The arc includes its endpoints X and Y. These two endpoints determine two arcs.

12. Find the volume of the inside of a pipe whose length is 80" and the radius of whose inside base circle is $\frac{1}{4}$". Use $\pi \approx 3.14$.

Answer:
15.70 cubic inches.

13.	Find the total surface area of a cylinder the radius of whose base circle is 10 centimeters if the altitude of the cylinder is 100 centimeters. Use $\pi \approx 3.1$.	Answer: 6820 square centimeters
14.	Find the amount of paper needed to make labels (without overlapping them) for 3 cans each of height 6" and of base circle radius 2". Use $\pi \approx 3.14$.	Answer: 226.08 square inches
15.	Find the volume of a cylindrical solid the diameter of whose base circle and whose height are each 3 meters. Leave your answer in terms of π.	Answer: $\frac{27}{4}\pi$ or 6.75π cubic meters.

Chapter 16

STATISTICS AND GRAPHS

This chapter is included as an attempt to accomplish the following objectives:

1. To develop some facility in the elementary uses of statistics through a study of some data collected by the government, by business, and by the students themselves.
2. To develop an appreciation of the important use of mathematics in everyday reading, thereby motivating and strengthening both.
3. To illustrate how data may be tabulated and the information contained in the data represented by different kinds of graphs.
4. To give instruction in reading information from a graph for which no table of data is given.
5. To develop some skill in finding measures of central tendency (modes, arithmetic means, and medians.)

The National Industrial Conference Board, 460 Park Avenue, New York 22, will supply good supplementary material for this chapter at your request.

A very informative and pleasantly written book on statistics is Darrell Huff's How to Lie With Statistics, W. W. Norton and Company, Inc., New York, New York.

A useful source of statistical information is the Statistical Abstract of the United States. This has been published annually since 1878. It is the standard summary of statistics on the industrial, social, political and economic organization of the United States and is prepared under the direction of the Bureau of the Census.

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Statistics has two main aspects. One is the organization, analysis and interpretation of a mass of collected data. The other consists of studying a total population by analyzing a sample of that population.

The use of statistical terminology in non-statistical literature is increasing and it is essential for the general reader to have a correct understanding of the most common terms and ideas.

16-1. Gathering Data.

The pupils might be asked to make tables of local statistics. There is the possibility that the local chamber of commerce could provide some interesting information. The school or city librarian could furnish statistics on numbers of volumes of various kinds. Prior to the study of this unit, pupils might be asked to keep records such as test scores, temperature, attendance, etc. Information of this nature provides excellent material for the graphs later in the chapter.

Answers to Exercises 16-1a

1. Boy O
2. 56 in.
3. 69 in.
4. Three
5. Two of each
6. Six
7. Eight
8. Seven
9. Seven

Answers to Exercises 16-1b

1. The enrollments seem to be increasing steadily.

2.

September	Increase
1955	
1956	125
1957	375
1958	125
1959	125
1960	50
1961	90
1962	110

3. a. 1956-1957 b. 1959-1960

Answers to Exercises 16-1c

1.

Year	Official Population	Population in Number of Ten Thousands	Increase in Ten Thousands (from previous column)	% of Increase (whole numbers of percent)
1890	88,243	9		
1900	122,931	12	3	33%
1910	204,354	20	8	67%
1920	334,162	33	13	65%
1930	435,573	44	11	33%
1940	499,261	50	6	14%
1950	749,587	75	25	50%
1960	1,302,161	130	55	73%

2. 1950-1960

3. 1950-1960

4. 1940-1950

5. No. The percent of increase depends on the base which is increased as well as the size of the increase.

*6. Answers will differ.
*7. Answers will differ.

16-2. Bar Graphs.

The materials needed will be graph paper for the pupils and a graph blackboard chart for demonstration as well as pupil use. If there is no graph blackboard chart available, many teachers use a music liner in making blackboard graphs. A few sweeps of the liner horizontally and vertically make a fine set of guidelines for graphing.

If students are encouraged to bring to class graphs from newspapers and magazines, they will make good bulletin board material and they will furnish illustrations of good and bad features of graphing.

Students are always surprised to find how often a graph is used as background for the humor in a cartoon. Some students even make collections of such cartoons.

Bar graphs may be the simplest graphs for pupils to understand. Choosing appropriate units may be difficult, and some time should be spent on this. In exercises requiring bar graphs, usually only one is shown. Either a horizontal bar graph or a vertical bar graph is acceptable.

Answers to Exercises 16-2a

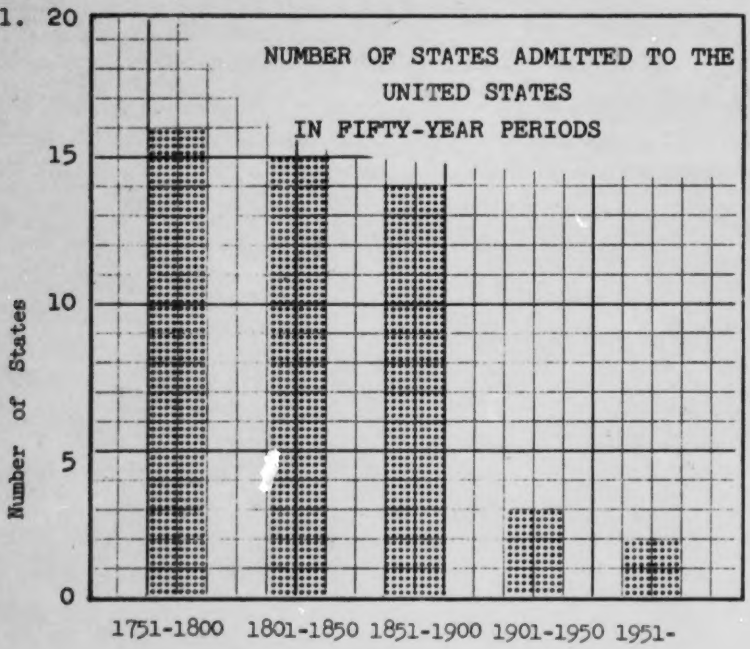
1. Alaska
2. About twice the size of Texas, about four times the size of California or Montana, and about five times the size of New Mexico or Arizona.
3. Over twice as large
4. Many comparisons can be made.

Answers to Exercises 16-2b

1. The title
2. They represent ten thousands of square miles.
   15 represents 15 x 10,000 or 150,000, etc.
3. Zero
4. 30 x 10,000 or 300,000
   60 x 10,000 or 600,000
5. The length of the bars
6. Two units
7. Two units
8. a. No b. In order of size, starting with the largest.

Answers to Exercises 16-2c

1.



Source: THE WORLD ALMANAC, 1962.

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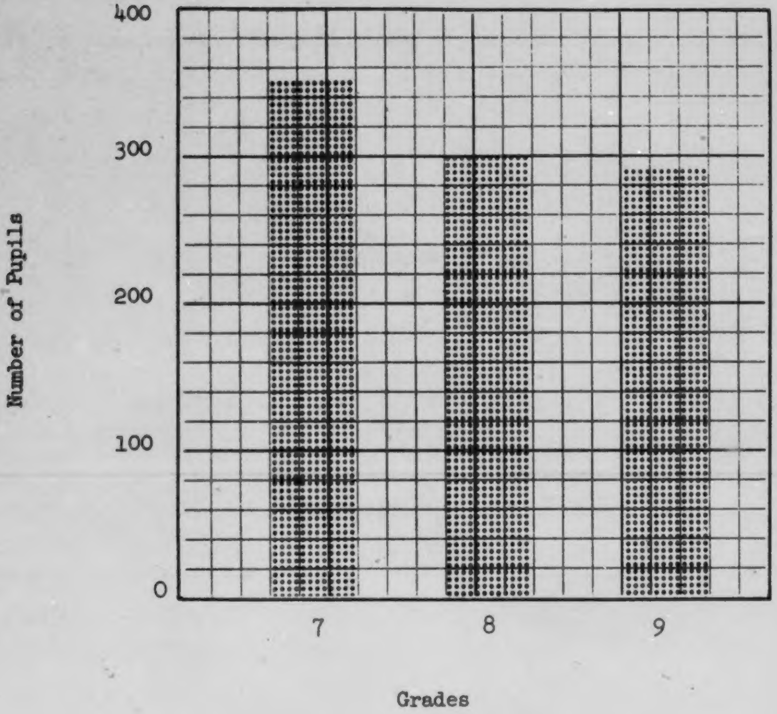
2.

ENROLLMENTS IN HOOVER JUNIOR HIGH SCHOOL



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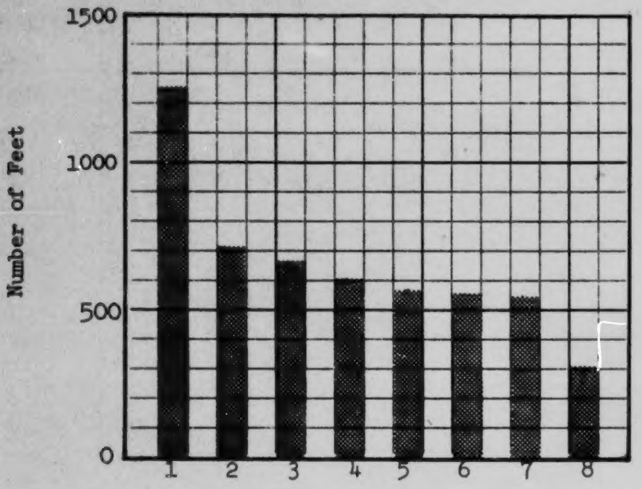
3. TALL STRUCTURES IN THE UNITED STATES



Source: THE WORLD ALMANAC, 1962

Key:

1. Empire State Building, New York
2. Terminal Tower, Cleveland
3. Space Needle, Seattle World's Fair
4. Prudential Building, Chicago
5. San Jacinto Monument, Houston
6. Washington Monument, Washington, D.C.
7. City Hall, Philadelphia
8. Statue of Liberty, New York

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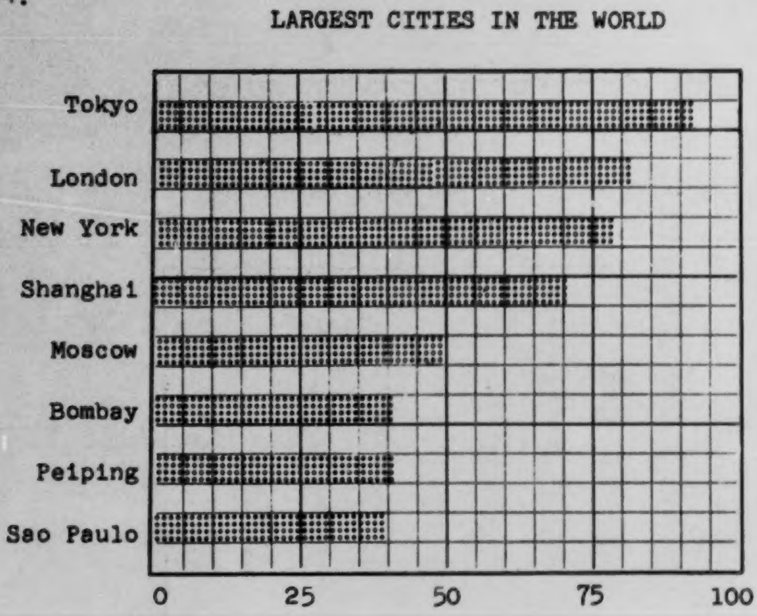
4.



Population in Hundred Thousands

Source: THE WORLD ALMANAC, 1962.

16-3. Broken-Line Graphs.

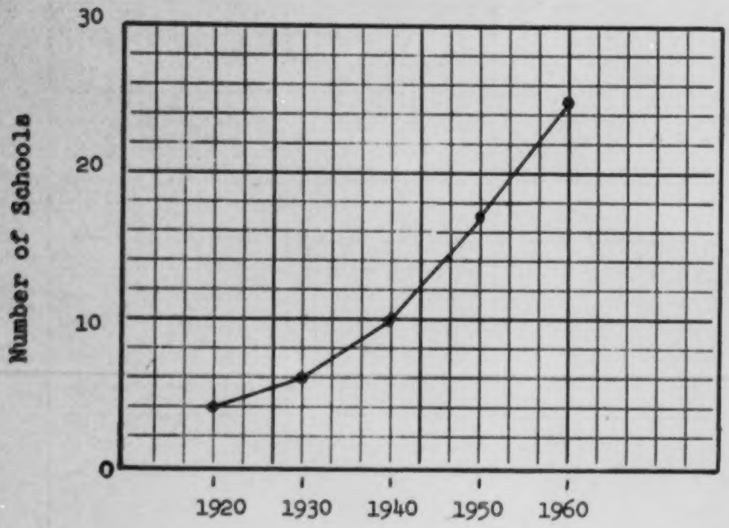
Answers to Exercises 16-3a

1. 40°, 36°, 60°, 52°, 28°.
2. Tuesday
3. Wednesday
4. Between Tuesday and Wednesday.
5. Between Thursday and Friday.
6. No, but a reasonable estimate could be made.

Answers to Exercises 16-3b

1.

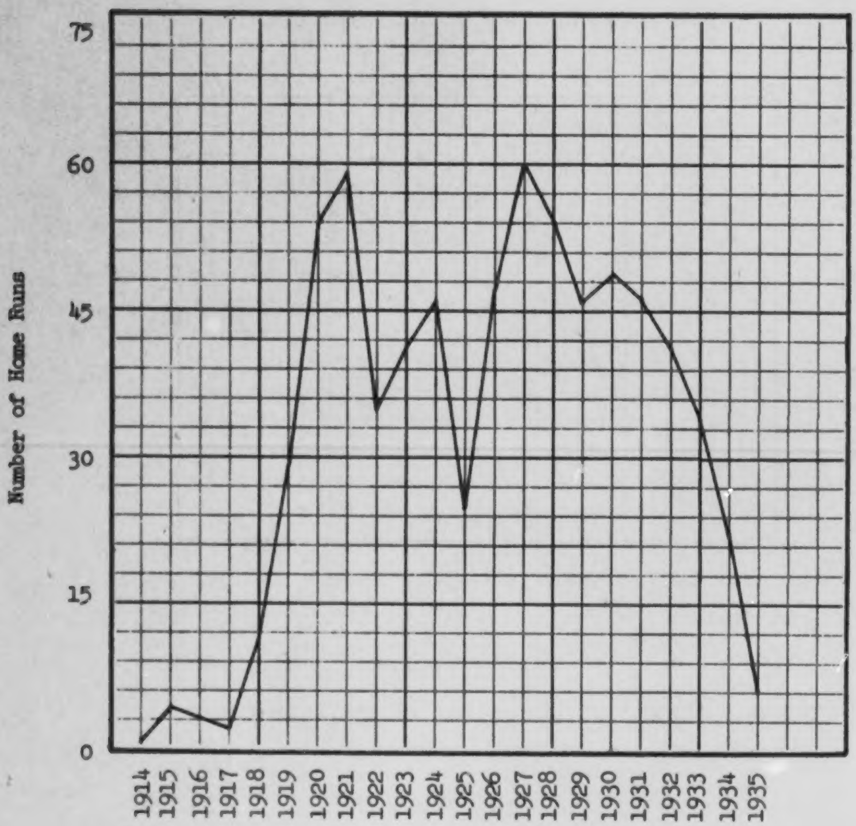
GROWTH OF SCHOOLS IN MADISON COUNTY



Source: Board of Education of Madison County

2.

HOME RUN RECORD OF BABE RUTH

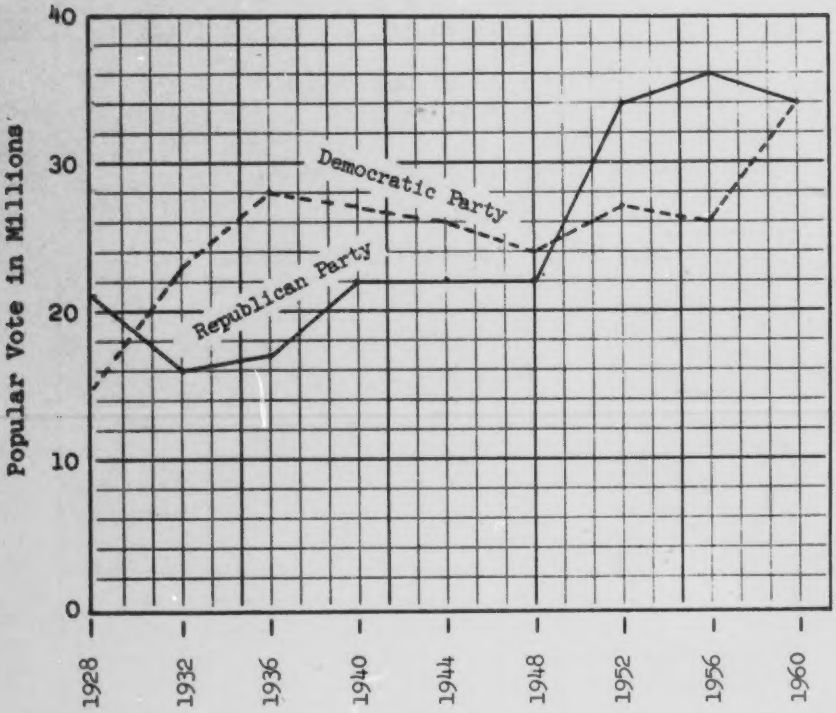


Source: THE WORLD ALMANAC, 1962

3.

POPULAR VOTE CAST FOR PRESIDENTIAL CANDIDATES

1928-1960



Source: THE WORLD ALMANAC, 1962

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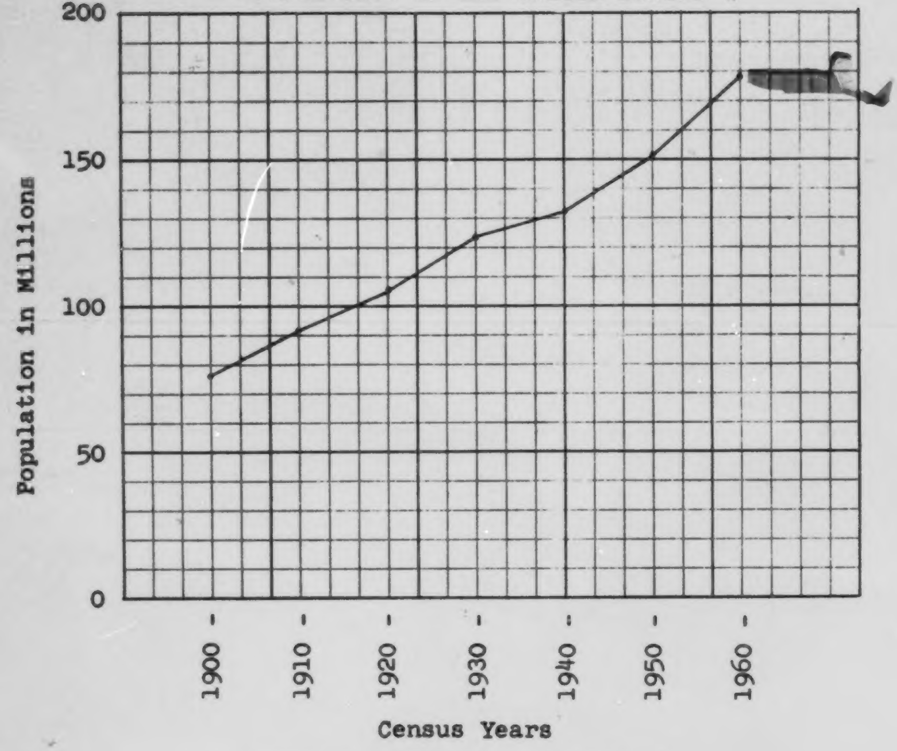
4. POPULATION OF THE UNITED STATES

Census Years	Population	Population in Millions
1900	75,994,575	76
1910	91,972,266	92
1920	105,710,620	106
1930	122,775,046	123
1940	131,669,275	132
1950	150,697,361	151
1960	179,323,175	179

Source: Bureau of the Census.

POPULATION OF THE UNITED STATES



Source: Bureau of the Census

16-4. Other Kinds of Graphs.

Since graphs are useful in conveying information, and are found so widely in newspapers and magazines, it is important that pupils become familiar with the general types which they will encounter.

Circle graphs are found frequently and lend themselves well to comparing parts with a whole. The questions on circle graphs will give some opportunity for practice in computation.

Rectangular bar graphs are often used in showing how a whole is distributed. It is not necessary for students to make such graphs in order to learn how to read them.

The pictograph makes a bar graph more interesting to read. However, care should be taken to use symbols which can be divided up easily to show fractional parts of units.

Answers to Exercises 16-4

1. a. The amounts are the same.
   b. Three times as much
   c. Food, \$1920; Rent, \$1920; Clothing, \$1280; Savings, \$640; Miscellaneous, \$640.

2. a. Education, 104.4°; Streets and parks, 72°; Police and fire protection, 90°; Welfare, 57.6°; Other, 36°.
   b. (1) $1\frac{1}{4}$ times as much.
      (2) $1\frac{4}{25}$ or about $1\frac{1}{5}$ times as much.
   c.

Education	\$ 926,260
Police and fire protection	798,500
Streets and parks	638,800
Welfare	511,040
Other	319,400
	3,194,000

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3. a. A -- 6 pupils
B -- 10
C -- 16
D -- 6
F -- 2

b. (1) 3 times as many
(2) $1\frac{3}{5}$ times as many
(3) Same

c. (1) Shows how the whole of anything is distributed.
(2) Provides for making comparisons easily.

4. a. $5,000,000,000

b. 1940 -- $5,000,000,000
1945 -- 45,000,000,000
1950 -- 40,000,000,000
1955 -- 60,000,000,000
1960 -- 80,000,000,000

c. Twice as much; 16 times as much.

5. a. (1) 0 (2) 20

b. $\frac{30}{22}$ or $1\frac{4}{11}$

c. $\frac{5}{1}$ or 5

d. Glancing at Figure 2, one might jump to the incorrect conclusion that State A spends five times as much on highways as State D, twice as much as State B, and more than three times as much as State C.

16-5. Averages.

The measures of central tendency are presented here. They are the arithmetic mean, the median and the mode. All three measures are averages. There are certain advantages and limitations to each. The nature of the particular distribution of measures as well as the purpose for which the average is chosen, will determine which of these averages should be used.

The mean or arithmetic mean is frequently called the average. A few very large or very small numbers can change a mean considerably.

The median is not difficult to find, but only tells that half of the numbers are below it and half are above it.

The mode is an interesting figure, but tells us nothing about the distribution of the data.

Answers to Exercises 16-5a

1. a. 74 b. 1995
2. 196
3. 159
4. 17,464 sq. ft.

Questions in Text

Mode

65 occurs more than any other number in the table. If there had been another 21, 21 would have been the mode.

If pupil L were 59 inches tall, there would be two modes, 65 and 59.

Range. Jim seems to be more consistent.

Answers to Exercises 16-5b

1. a. **Scores**

94
91
87
85
85
85
83
81
79
74
69
Total 913

b. Mode = 85

c. Median = 85

d. Range = 94 - 69 = 25

e. $\frac{913}{11} = 83 =$ the mean

2. a. Mean = \$6100.

b. Three

c. Seven

d. No. See (f).

e. If there is an even number (2n) of items (as there are 10 in this problem), the median is the average of the nth and (n + 1)th items

$$\frac{\$5000 + \$5500}{2} = \$5250.$$

f. Median is better than the mean, since the mean gives the impression that the salaries are higher than they are. The mean is affected by the large salary of \$12,500, but the median is not.

g. (1) No. (2) Yes (3) No.

h. The median is useful because it is not affected by one or even a very few inordinately large scores.

3. a. 56.5 b. 49 c. 38

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*16-6. Grouping Data.

Answers to Exercises 16-6a

1. 7

2. a. The 17th pupil
   b. 62

3. 62

Answers to Exercises 16-6b

1. Such information as ages, marks, attendance.

2. a. The age group 15-17
   b. 16

3.

Groups	Tallies	Frequency
50--54	𝍸	5
55--59	𝍷𝍷𝍷	3
60--64	𝍷𝍷𝍷𝍷	4
65--69	𝍷𝍷𝍷𝍷	4
70--74	𝍸 𝍷	6
75--79	𝍷𝍷𝍷	3
80--84	𝍷𝍷𝍷𝍷	4
85--89	𝍷	1
90--94	𝍷	1
		31

The median, or the 16th item is in the 65-69 group.

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16-8. Chapter Review.

Answers to Exercises 16-8

1. a. Swimming is the most popular sport, and the others, in order, are horseback riding, canoeing, archery, tennis.

   b. 1 unit = 4 girls.

   c.

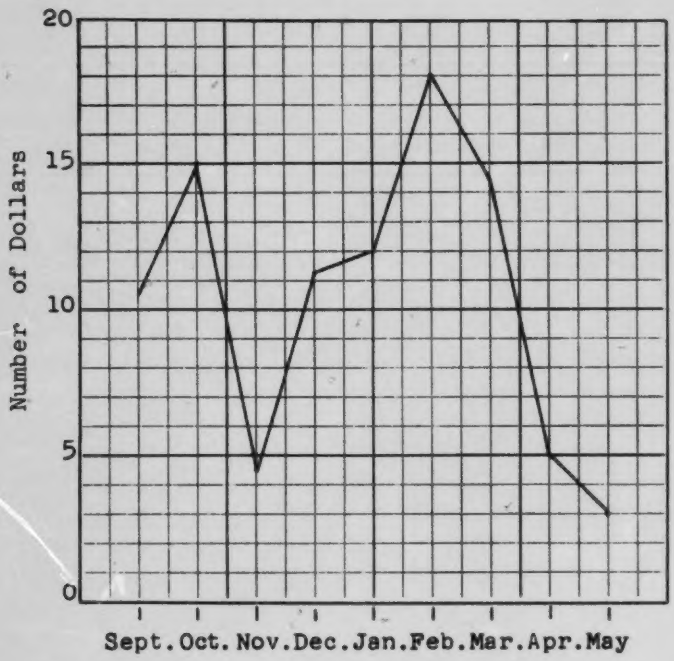
Swimming	78
Horseback riding	40
Canoeing	36
Archery	22
Tennis	20

   d. Swimming

   e. (1) More than twice
      (2) More than three times
      (3) About the same

2.

JIM'S MONTHLY EARNINGS
DURING A SCHOOL YEAR



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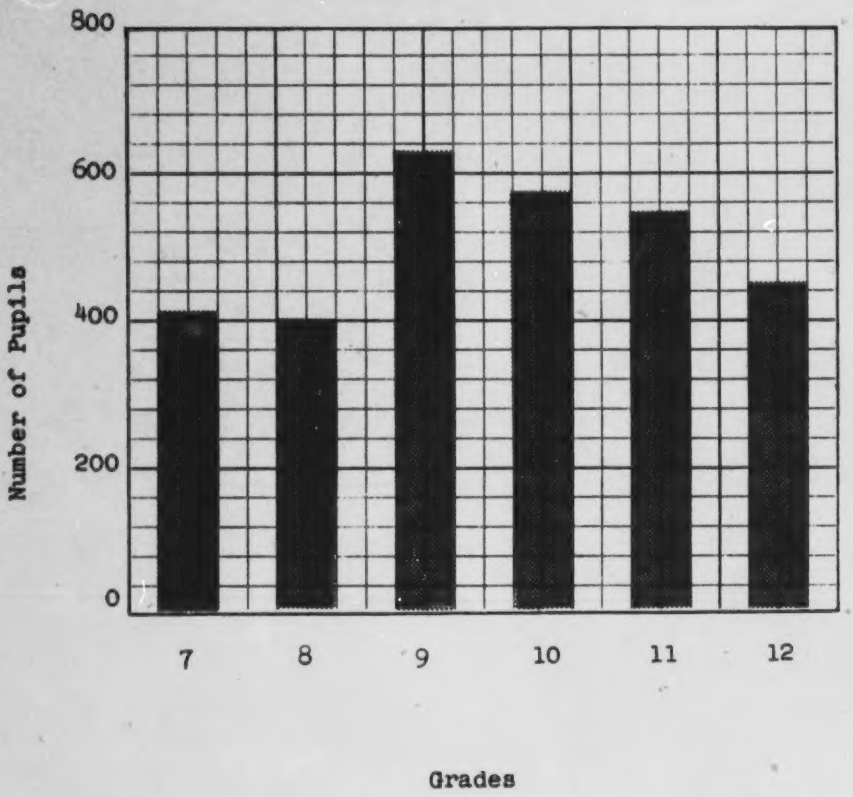
3. a. 5000 b. 15000 c. 3000 d. 155,000

4.

ENROLLMENT IN ROOSEVELT HIGH SCHOOL



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5. a. Scores:

97
95
94
92
90
89
86
85
85
83
81
72
71
70
62

Total 1252

Range:
97-62 = 35

b. Mode = 85

c. Mean = $\frac{1252}{15} \approx 83.5$

d. Median = 85

e. No. Yes.

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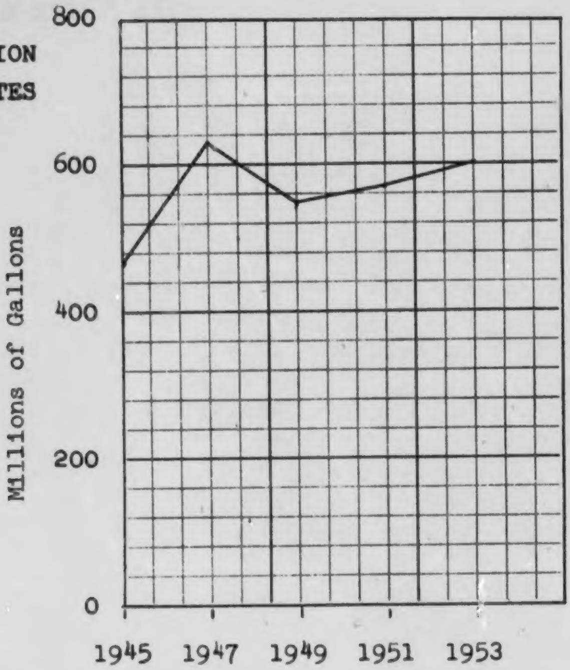
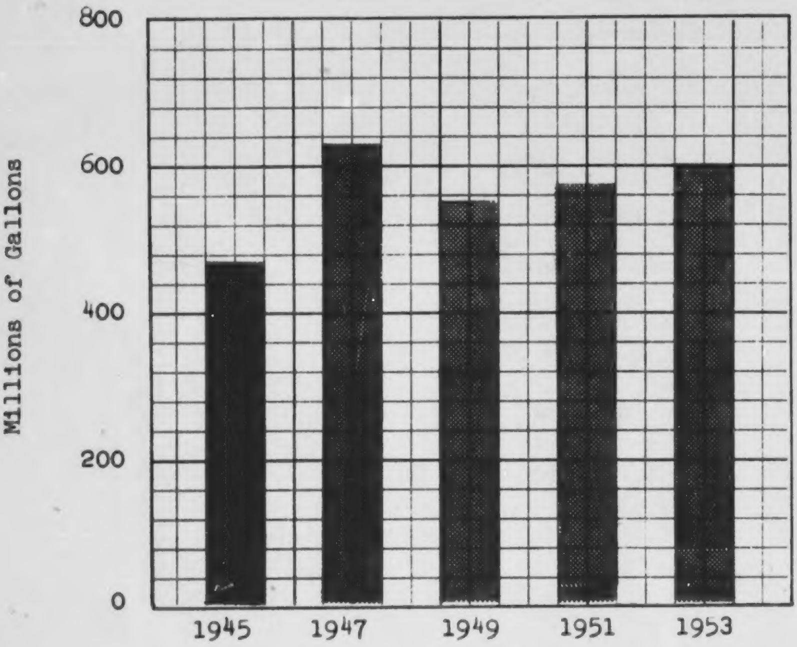
*6.

ICE CREAM PRODUCTION
IN THE UNITED STATES





16-9. Cumulative Review.

Answers to Exercises 16-9

1. 2
2. 4
3. $\{6, 8, 10\}$
4. $\frac{2}{5}$
5. 4
6. $\frac{9}{2}$, or $4\frac{1}{2}$, $\frac{18}{4}$, $\frac{27}{6}$, etc.
7. Check to see if $ad = bc$.
8. $\frac{1}{4}$ inch
9. 10.5
10. $(\frac{7}{8})^6$ or $\frac{7^6}{8^6}$
11. 58
12. Mode
13. 15 inches.
14. a. $b = 45$
    b. $b = 40$
15. 14.5 hours.
16. $m\angle X = 130$
17. a. 7.016
    b. 405.07
18. a. 57,500
    b. 686,000
    c. 15.04
    d. 0.1

Sample Test Questions for Chapter 16

This set of questions is not intended as a chapter test. Teachers should construct a chapter test carefully by combining selected items from this set of questions and questions of their own writing. Care should be exercised to avoid making the test too long.

In Problems 1-8, each description in the left column defines one term on the right. Write the letter of the term on the right beside the description which it matches.

Matching:

(g) 1. The difference between the largest and smallest number in a set.

(a) 2. The middle number when data are ordered.

(b) 3. The number occuring most often in a list of data.

(c) 4. The sum of all the numbers in a set divided by the number of numbers in the set.

(d) 5. A graph used to show change in some item.

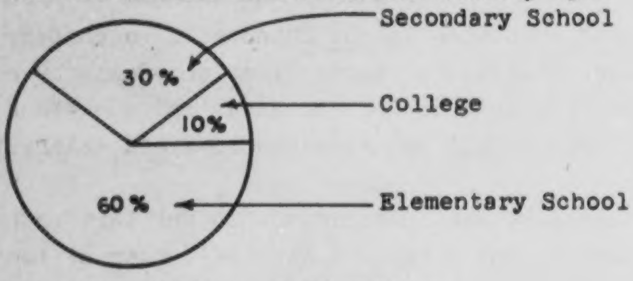
(i) 6. A graph used to show comparison between similar items.

(f) 7. A graph used to show comparison between parts of a whole and between the whole and any of its parts.

(h) 8. A graph which uses symbols or pictures to represent numbers.

a. Median
b. Mode
c. Arithmetic Mean
d. Broken-Line Graph
e. Average
f. Circle Graph
g. Range
h. Pictograph
i. Bar Graph

9. School Enrollment in the United States, 1960



Source: U.S. Office of Education

$\frac{3}{5}$ a. What fractional part of the total enrollment is in elementary school?

$\frac{3}{10}$ b. What fractional part of the total enrollment is in secondary school?

$\frac{1}{10}$ c. What fractional part of the total enrollment is in college?

36 d. What is the measure of the central angle which indicates college enrollment?

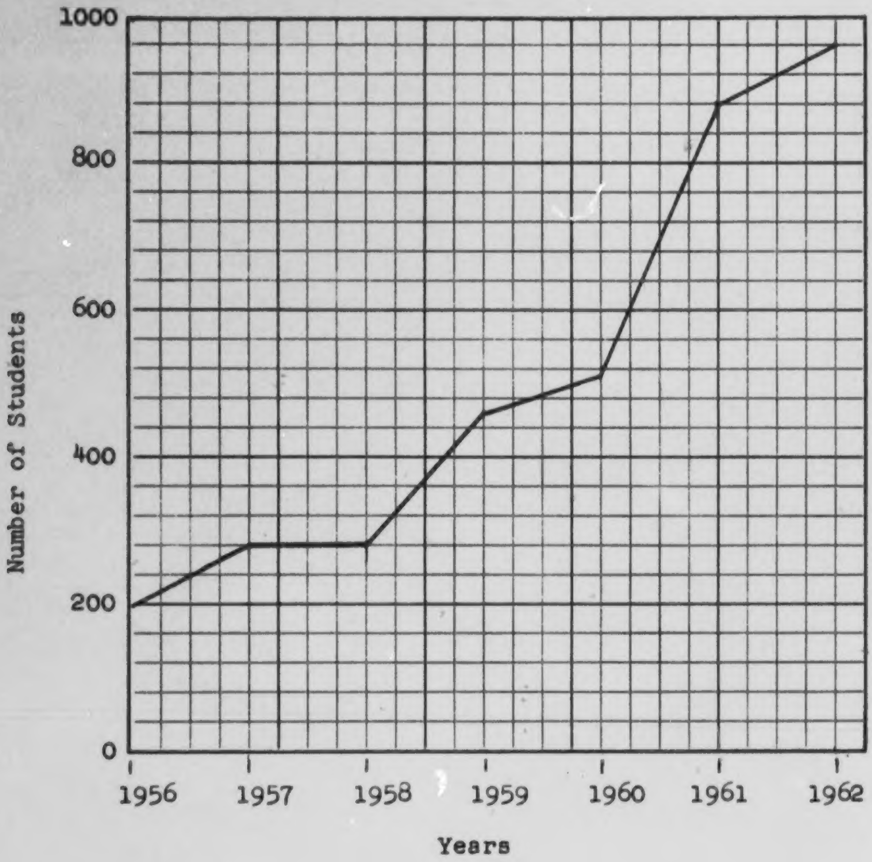
If there were about 40,000,000 students enrolled in school in 1960,

24,000,000 e. How many were in elementary school?

12,000,000 f. How many were in secondary school?

4,000,000 g. How many were in college?

10. **MADISON JUNIOR HIGH SCHOOL ENROLLMENT IN SEPTEMBER OF EACH YEAR**



a. What kind of graph is the graph in this problem? (Broken-line graph)

b. What was the enrollment in 1962? (960)

c. In what two years was the enrollment the same? (1957,1958)

d. During which year did the enrollment increase the most? (1960-61)

e. What is the general trend of the enrollment? (increasing)

f. What is the difference in enrollment between 1956 and 1961? (680)

g. What was the percent of increase in enrollment in 1956-1957? (40%)

Chapter 17

NEGATIVE RATIONAL NUMBERS

17-1. Introduction.

The material of this section is motivational. Students are often suspicious of negative numbers. The reason for this suspicion is seen in this problem:

> If Jimmie has 5 marbles and he gives 3 to Eddie and 4 to Dudley, how many does he have left?

The student will say that this is impossible. He is correct in saying this; this distribution of marbles cannot be carried out. The student will not be satisfied with $^{-}2$ as an answer.

An effort is made to lead the student to the negative numbers along familiar paths. The thermometer provides a familiar example in which we have two number scales on the same line extending in opposite directions from a common origin.

It is carefully pointed out that a succession of decreases in temperature will eventually carry us over onto the other side of the origin. This idea is repeated with an example from business. Armed with these examples we are ready in the next section to call numbers on one side of the origin positive and numbers on the other side negative.

Students cannot properly appreciate the importance of negative numbers until they have had a course in Algebra. Chapter 19 of this text on coordinates in the plane should, however, help to give the student some confidence in the usefulness of these numbers.

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17-2. The Negative Half-Line.

In this section the negative rational numbers are located on the number line and named. Some explanation of the raised "minus sign" used in naming these numbers, as in $^{-}3$ or $^{-}2.6$, is in order.

The "minus sign" is used in three ways in elementary mathematics:

i) as part of the name of a negative number;
ii) to denote the binary operation of subtraction;
iii) to denote the unary operation of finding the "opposite" or "additive inverse."

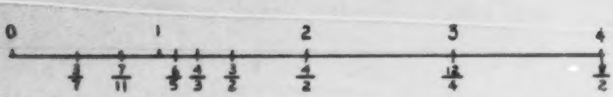
These three meanings are all quite different. The student is usually confused about which meaning is intended at the start of his study of negative numbers. His confusion finally disappears when he learns enough to appreciate the consistency between the three uses of the minus sign.

The raised minus sign is used in an effort to avoid any confusion on this score. In this text the minus sign in its usual position is used only to denote the operation of subtraction (meaning ii above). The raised minus sign is used to provide a common name for negative numbers (meaning i above). The operation of finding the opposite or additive inverse (meaning iii above) is discussed in this section and throughout the chapter but no symbol has been introduced to designate this operation.

In accordance with this convention the raised dash is used only in front of common names for rational numbers (that is proper or improper fractions in simplest form, mixed numbers, or decimals) to provide common names for negative numbers. We should not write expressions such as "$^{-}(3 + 2)$" or "$^{-}\left(\frac{\frac{14}{3}}{\frac{7}{5}}\right)$". In fact we should not even write expressions like "^{-}a" or "^{-}x" even when we specify that a or x represent positive numbers. Realizing this we have nevertheless used the notation "^{-}a" and "^{-}b" where it was very difficult to avoid.

Answers to Exercises 17-2a

1.


0
1
2
3
4
$\frac{3}{7}$
$\frac{7}{11}$
$\frac{6}{5}$
$\frac{4}{3}$
$\frac{3}{2}$
$\frac{4}{2}$
$\frac{12}{4}$
$\frac{8}{2}$

2. a. $\frac{4}{2} > \frac{4}{3}$ b. $\frac{3}{2} > \frac{6}{5}$ c. $\frac{8}{2} > \frac{3}{7}$ d. $\frac{7}{4} < \frac{12}{4}$

3. a. $4 \cdot 3 > 2 \cdot 4$ c. $8 \cdot 7 > 2 \cdot 3$

   b. $3 \cdot 5 > 2 \cdot 6$ d. $7 \cdot 4 < 12 \cdot 4$

Answers to Exercises 17-2b
(Class Discussion)

1. a. negative e. negative
   b. positive f. positive
   c. positive g. negative
   d. negative h. positive

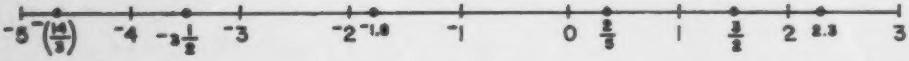
2. a. The set of positive rational numbers.
   b. The set of negative rational numbers.
   c. The set of positive integers.
   d. The set of negative integers.

3. a. ${}^-4$ d. $2\frac{1}{2}$ g. 6.4
   b. 8 e. $\frac{5}{4}$ h. 0
   c. ${}^-(\frac{2}{3})$ f. ${}^-5.3$

4. ${}^-6$, ${}^-4$, ${}^-(\frac{7}{4})$, ${}^-(\frac{3}{8})$, $\frac{1}{4}$, $\frac{5}{8}$, $\frac{3}{4}$; $\frac{3}{4}$ is the greatest, ${}^-6$ is the least.

Answers to Exercises 17-2c

1.


${}^-5$
${}^-(\frac{14}{3})$
${}^-4$
$-3\frac{1}{2}$
${}^-3$
${}^-2$
${}^-1.8$
${}^-1$
0
$\frac{2}{3}$
1
$\frac{3}{2}$
2
2.3
3

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2. a. $^-3\frac{1}{2} < \frac{3}{2}$

   b. $\frac{2}{5} < \frac{3}{2}$

   c. $^-3\frac{1}{2} < {}^-1.8$

   d. $^-3\frac{1}{2} > {}^-(\frac{14}{3})$

   e. $\frac{2}{5} > {}^-1.8$

   f. $^-1.8 < 2.3$

3. a. D b. F c. B d. A e. C f. E

17-3. Addition on the Number Line.

The number line is the basis for explaining the operation of addition. Students are familiar with this use of the number line from Chapter 8. There should be no difficulty in showing that the sum of two negative numbers is the opposite of the sum of the absolute value of these numbers. However the term, "absolute value" is not used.

It is more difficult for students to see the result of adding a positive and a negative number. The sum is the difference of the absolute values of the numbers and is positive if the number with the larger absolute value is positive and negative if this number is negative.

The method of adding rational numbers is summarized only informally. The primary reason for this is that an accurately stated rule requires the use of absolute value. The development of absolute value is left for the work in the ninth grade. Many students will have an intuitive concept of this idea, however. In one exercise the students are asked to state a method of addition. Answers should be judged on the insight into the operation rather than its precise statement. The number line may creep into the students' explanation even though the directions ask them to think about addition without the help of the number line. Any answer that shows understanding should be accepted.

Teaching Suggestions: The time required to draw number lines discourages students. It is helpful if the teacher duplicates number lines or uses graph paper, because the time required to do the assignment is decreased, and the students' attention is concentrated on the addition rather than the drawing of lines. Since sheets of number lines will be useful in other sections of this chapter, quite a few copies should be made.

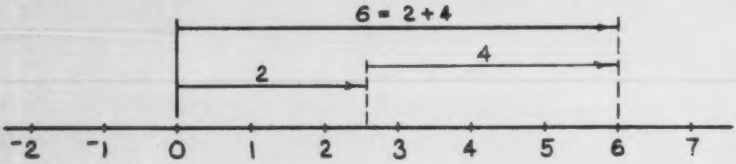
After reading the explanation in the text, the class members should do one addition problem of each type before attempting the homework. Give the problem, using small numbers. Allow time for most students to finish, then do it correctly on the board or let a student do so. Each pupil should do the problems; it is not adequate to watch someone else.

Answers to Exercises 17-3a

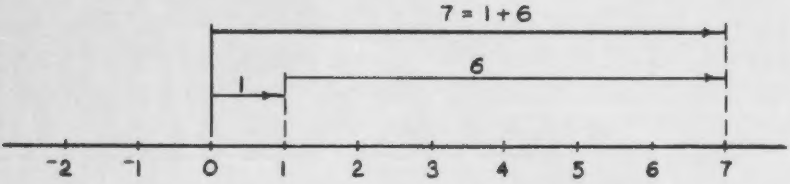
1. Right
2. Left
3. All arrows point to the right; the second arrow starts at the head of the first arrow. The answer arrow starts at zero and goes to the head of the second arrow.
4. All arrows point to the left. The arrangement is the same as in Problem 3.
5. The first arrow starts at zero. It may go right or left. The second arrow starts at the head of the first arrow; it goes in the opposite direction. The answer arrow starts at zero and goes to the head of the second arrow. The answer arrow may point in either direction.
6. Use an arrow for each addend. Each arrow is slightly above the preceding arrow. The answer arrow runs from zero to the head of the last arrow.
7. An arrow for 3. A dot may be used for zero and an answer arrow added for emphasis.

Answers to Exercises 17-3b

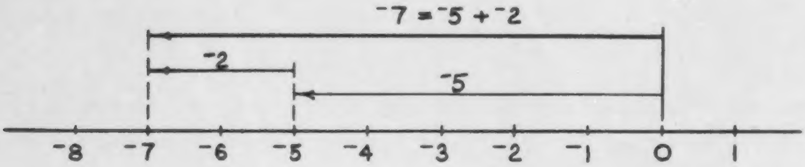
1. a.


6 = 2 + 4
4
2
-2 -1 0 1 2 3 4 5 6 7

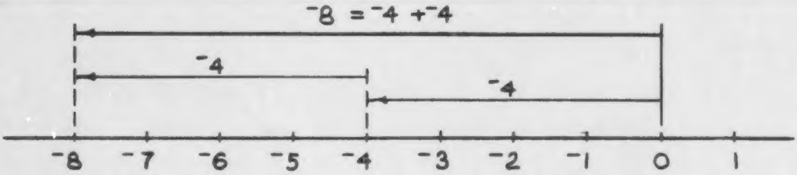
b.


7 = 1 + 6
6
1
-2 -1 0 1 2 3 4 5 6 7

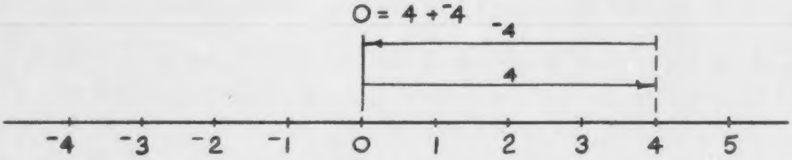
c.


-7 = -5 + -2
-2
-5
-8 -7 -6 -5 -4 -3 -2 -1 0 1

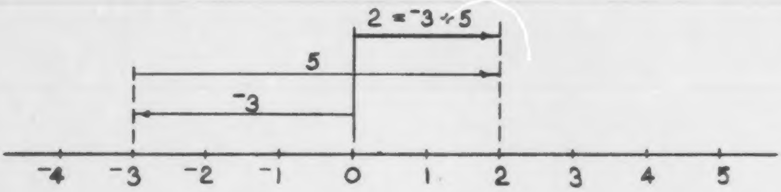
d.


-8 = -4 + -4
-4
-4
-8 -7 -6 -5 -4 -3 -2 -1 0 1

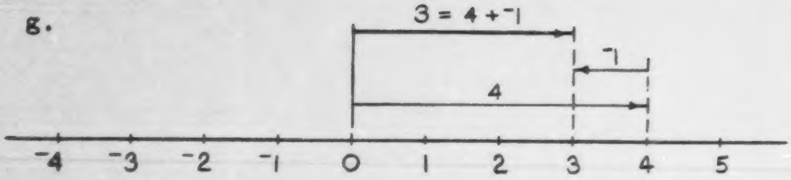
e.


0 = 4 + -4
-4
4
-4 -3 -2 -1 0 1 2 3 4 5

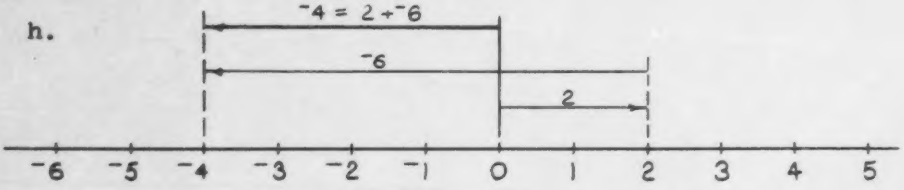
f.


2 = -3 + 5
5
-3
-4 -3 -2 -1 0 1 2 3 4 5

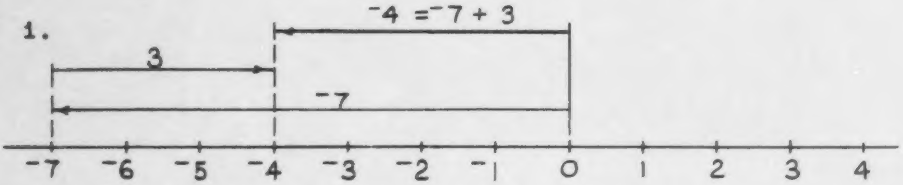
g.


3 = 4 + ˉ1
ˉ1
4
ˉ4 ˉ3 ˉ2 ˉ1 0 1 2 3 4 5

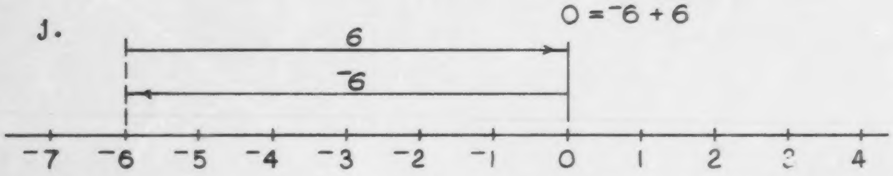
h.


ˉ4 = 2 + ˉ6
ˉ6
2
ˉ6 ˉ5 ˉ4 ˉ3 ˉ2 ˉ1 0 1 2 3 4 5

i.


ˉ4 = ˉ7 + 3
3
ˉ7
ˉ7 ˉ6 ˉ5 ˉ4 ˉ3 ˉ2 ˉ1 0 1 2 3 4

j.


0 = ˉ6 + 6
6
ˉ6
ˉ7 ˉ6 ˉ5 ˉ4 ˉ3 ˉ2 ˉ1 0 1 2 3 4

2. a. 0
   b. 4
   c. $^{-}6$
   d. $^{-}1$
   e. 12
   f. $\frac{2}{3}$
   g. $\frac{1}{2}$
   h. 0

3. a. $^{-}5$
   b. 3
   c. $\frac{3}{2}$
   d. $^{-}(\frac{3}{4})$
   e. 2
   f. $^{-}4$
   g. 25
   h. ^{-}a, the negative of the number a.

4. a. +
   b. -
   c. -
   d. +
   e. -
   f. +
   g. -
   h. +
   i. -
   j. +
   k. -

5. a.

January	5000	April	1000
February	2400	May	$^-3000$
March	$^-1000$	June	$^-2000$

   b. $2400
   c. $400

6. **BRAINBUSTER:**

1st boy: Taking 2 steps per second requires 14 seconds when the steps are not moving. During this time the escalator has moved half-way down. Boy requires 28 seconds when the escalator is moving. Since the boy goes up 2 steps while the escalator goes down 1 step, his actual rate is 1 step per second.

2nd boy: Goes up 3 steps while the escalator goes down 1 step. The boy's actual speed is 2 steps per second. Time required is 14 seconds.

Answers to Exercises 17-3c

1. a. $10\frac{2}{5}$
   b. $10\frac{2}{5}$
   c. 10
   d. 10
   e. $7\frac{1}{2}$

2. a. $^-4$
   b. $^-4\frac{2}{3}$
   c. $^-6\frac{1}{3}$
   d. $^-6\frac{1}{3}$
   e. $^-7\frac{1}{4}$
   f. $^-7\frac{1}{4}$
   g. $^-4\frac{1}{3}$

3. a. $^{-}1$

   b. $^{-}2$

   c. $^{-}1\frac{1}{5}$

   d. $^{-}1\frac{1}{5}$

   e. $1\frac{1}{5}$

   f. $5\frac{3}{4}$

   g. $5\frac{3}{4}$

   h. $^{-}5\frac{3}{4}$

   i. 0

4. The wording of these statements is less important than the understanding. Informal description of the addition process will be found in the text following the exercises.

Answers to Exercises 17-3d

1. a. Parts (a) and (b) of Problem 1, Parts (c) and (d) of Problem 2 and Parts (c) and (d) of Problem 3 illustrate the commutative property of addition.
   b. Parts (c) and (d) of Problem 1, and Parts (e) and (f) of Problem 2 illustrate the associative property of addition. Also Parts (f) and (g) of Problem 3.
   c. Part (e) of Problem 1 and Part (g) of Problem 2 illustrate the addition property of zero.

2. a. 0 b. 0 c. 0

3. The numbers in each part are opposites, or additive inverses. The sum is zero in each case.

4. Yes.

17-4. Subtraction of Rational Numbers.

There are several possibilities in using the number line to show subtraction of rational numbers. The method used in the text has some advantages. First, it shows the relation of addition and subtraction, namely, that $a + b = c$ and $c - a = b$ are two statements of the same relationship.

Second, there is no essential change from the method of picturing addition; the diagrams look the same but the answer is in a different location.

As in addition, there is no formal rule for the method used to subtract. There are exercises that bring out the fact that the same results are obtained by subtraction as are obtained by adding the additive inverse of the subtrahend. It is important for the student to develop a feeling for the operation, not a mechanical and meaningless manipulation.

A very important concept for the students to get is that with the extension of the number line to include negative numbers, we can now subtract any two rational numbers. In other words, we have a set of numbers that is closed under subtraction.

Teaching suggestions: In the pupils' later study of mathematics the operation of subtraction is not necessary. The addition of the opposite of a number gives the same result. When presenting subtraction stress the fact that $a + b = c$ and $c - a = b$ are two statements for the same relation. After going over the explanation in the text with the students, give some of the practice problems in Exercises 17-4a (Class Discussion) as written work. Check to see that the students understand the process and know the difference between addition and subtraction.

Again, the time-consuming task of making number lines discourages students and diverts their attention from the ideas of the lesson. The teacher is urged to provide duplicated number lines when assigning problems in which they are used.

After considerable practice in using the number line to find answers to subtraction problems, shorter methods might be discovered. The teacher should be sure that the pupil understands whatever method is used.

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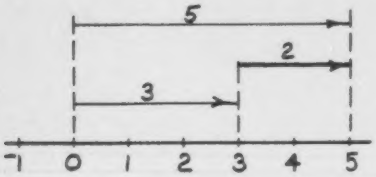
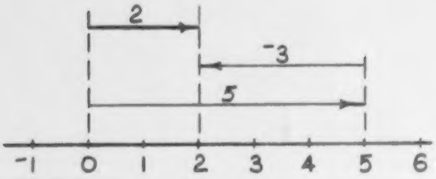
Answers to Exercises 17-4a

1. a. $4 - 7 = n$
   b. $n = {}^-3$
   c. $7 + ({}^-3) = 4$
2. a. ${}^-4 - 7 = n$
   b. $n = {}^-11$
   c. $7 + ({}^-11) = {}^-4$
3. a. ${}^-4 - ({}^-7) = n$
   b. $n = 3$
   c. ${}^-7 + 3 = {}^-4$
4. a. $4 - ({}^-7) = n$
   b. $n = 11$
   c. ${}^-7 + 11 = 4$
5. a.

$5 - 3 = 2$

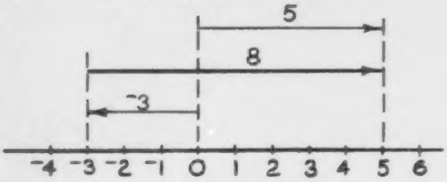

5
2
3
⁻1 0 1 2 3 4 5

$5 + ({}^-3) = 2$

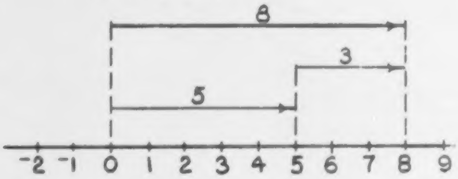

2
⁻3
5
⁻1 0 1 2 3 4 5 6

b.

$5 - ({}^-3) = 8$

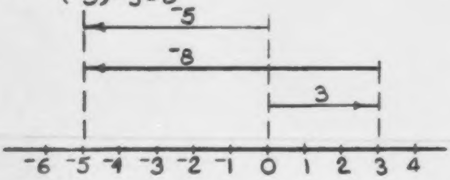

5
8
⁻3
⁻4 ⁻3 ⁻2 ⁻1 0 1 2 3 4 5 6

$5 + 3 = 8$

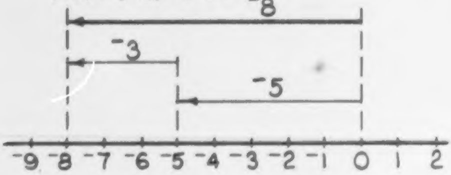

8
3
5
⁻2 ⁻1 0 1 2 3 4 5 6 7 8 9

c.

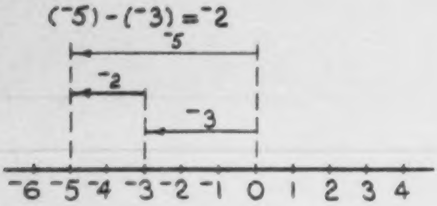
$({}^-5) - 3 = {}^-8$


⁻5
⁻8
3
⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 1 2 3 4

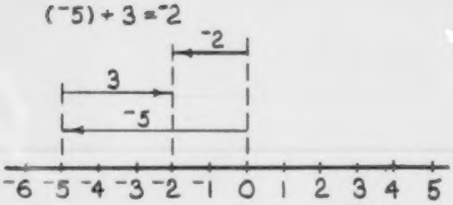
$({}^-5) + ({}^-3) = ({}^-8)$


⁻8
⁻3
⁻5
⁻9 ⁻8 ⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 1 2

d.



$(^{-}5) - (^{-}3) = {}^{-}2$
$^{-}5$
$^{-}2$
$^{-}3$
$^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 1 2 3 4



$(^{-}5) + 3 = {}^{-}2$
$^{-}2$
3
$^{-}5$
$^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 1 2 3 4 5

6. The result is the same when a number is subtracted and when the opposite of the number is added. Notice the diagrams are different.

In the text:

$1 - (^{-}2) = 1 + 2 = 3$

$1 - 3 = 1 + (^{-}3) = {}^{-}2$

Answers to Exercises 17-4b

1. a. $^{-}4 + (^{-}2) = {}^{-}6$
   b. $^{-}6 + 1 = {}^{-}5$
   c. $8 + 3 = 11$
   d. $1 + 0 = 1$
   e. $4 + (^{-}6) = {}^{-}2$
   f. $^{-}4 + (^{-}6) = {}^{-}10$
   g. $4 + 6 = 10$
   h. $0 + (^{-}1) = {}^{-}1$

2. a. 2
   b. 4
   c. $^{-}7$
   d. 10
   e. $^{-}10$
   f. $^{-}6$
   g. $^{-}11$
   h. 12
   i. 2
   j. 12
   k. $^{-}7$
   l. 11
   m. $^{-}7$
   n. 13

3. a. $^{-}10$
   b. 100
   c. $^{-}\frac{1}{2}$
   d. $^{-}\frac{7}{9}$
   e. $\frac{8}{5}$
   f. $\frac{49}{51}$

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4. a. 3 b. $^-13$ c. 13 d. $^-3$

5. a. 3 b. $^-13$ c. 13 d. $^-3$

6. Adding the additive inverse of a number gives the same result as subtracting the number.

After considerable practice in using the number line to find the answers in subtraction problems, pupils may devise shorter methods of arriving at the answers.

7. a. $x = {}^-3$

b. $x = 4$

c. $x = {}^-11$

d. $x = 15$

e. $x = {}^-(\frac{13}{6})$ or $^-2\frac{1}{6}$

f. $x = 14$

g. $x = \frac{1}{6}$

h. $x = {}^-5$

17-5. Multiplication of Rational Numbers.

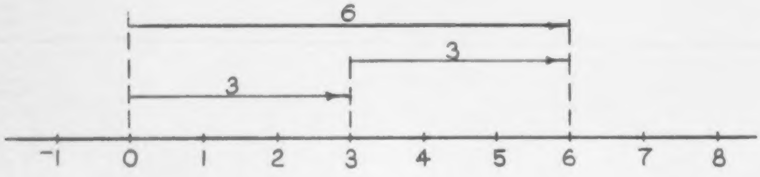
The multiplication of a positive and a negative number is developed in this section.

The product of a negative number multiplied by a positive number is easy to demonstrate on the number line. On the other hand, the interpretation of multiplying a positive number by a negative number has little meaning. The commutative property is used to make such products meaningful. There is occasion to show that the associative property holds also.

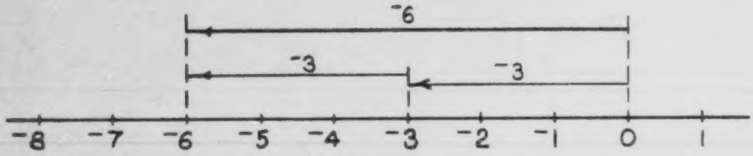
While the product of $3 \cdot ({}^-2)$ is the same as the product of $2 \cdot ({}^-3)$, this is not an illustration of the commutative property; both factors are different. Students should be aware of the difference.

Answers to Exercises 17-5a

1.



2.


$^{-}6$
$^{-}3$
$^{-}3$
$^{-}8$ $^{-}7$ $^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 1

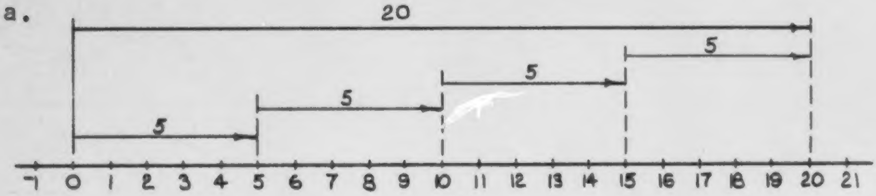
3. The commutative property for multiplication.

4. No. The product is the same but the factors are different.

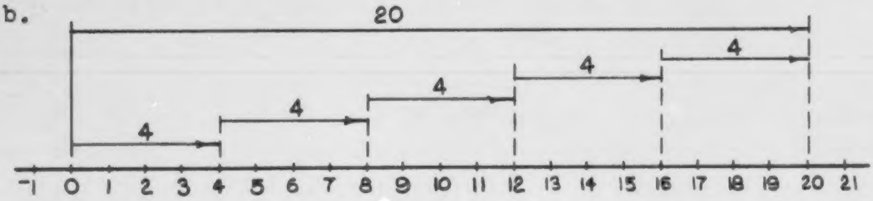
5. $^{-}6$

Answers to Exercises 17-5b

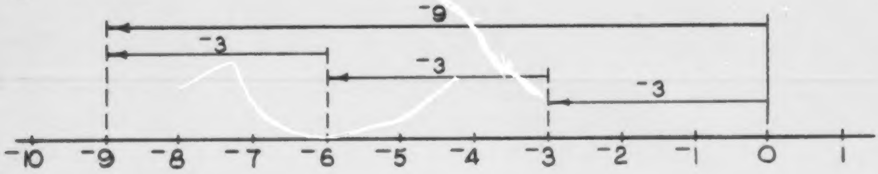
1. a.


20
5
5
5
5
$^{-}1$ 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

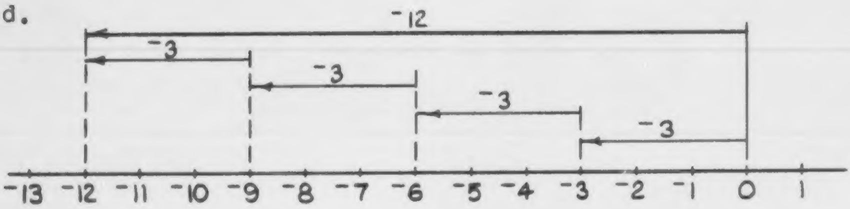
b.


20
4
4
4
4
4
$^{-}1$ 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

c.


$^{-}9$
$^{-}3$
$^{-}3$
$^{-}3$
$^{-}10$ $^{-}9$ $^{-}8$ $^{-}7$ $^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 1

d.


$^{-}12$
$^{-}3$
$^{-}3$
$^{-}3$
$^{-}3$
$^{-}13$ $^{-}12$ $^{-}11$ $^{-}10$ $^{-}9$ $^{-}8$ $^{-}7$ $^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 1

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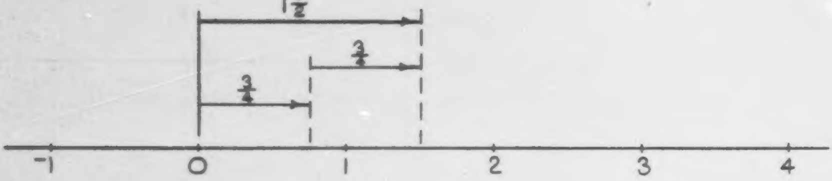
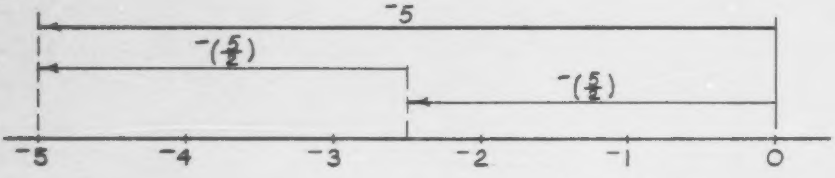
e.



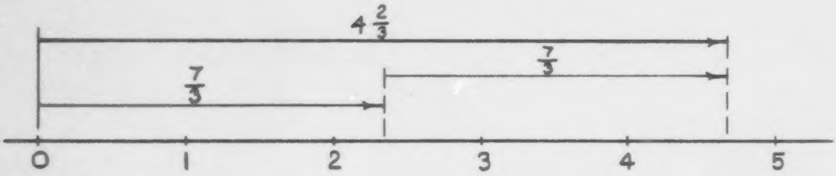
$1\frac{1}{2}$
$\frac{3}{4}$
$\frac{3}{4}$
$^{-}1$
0
1
2
3
4

f.



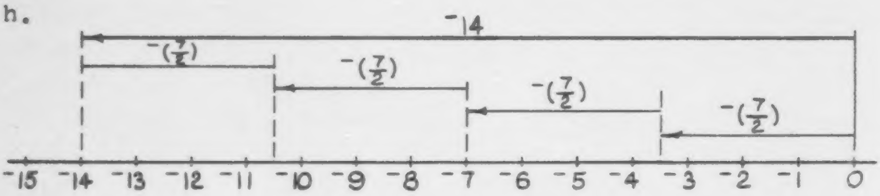
$^{-}5$
$^{-}(\frac{5}{2})$
$^{-}(\frac{5}{2})$
$^{-}5$
$^{-}4$
$^{-}3$
$^{-}2$
$^{-}1$
0

g.



$4\frac{2}{3}$
$\frac{7}{3}$
$\frac{7}{3}$
0
1
2
3
4
5

h.



$^{-}14$
$^{-}(\frac{7}{2})$
$^{-}(\frac{7}{2})$
$^{-}(\frac{7}{2})$
$^{-}(\frac{7}{2})$
$^{-}15$
$^{-}14$
$^{-}13$
$^{-}12$
$^{-}11$
$^{-}10$
$^{-}9$
$^{-}8$
$^{-}7$
$^{-}6$
$^{-}5$
$^{-}4$
$^{-}3$
$^{-}2$
$^{-}1$
0

2. a. $^{-}9$ c. $^{-}5$
   b. $^{-}12$ d. $^{-}14$

3. a. Positive
   b. Negative

4. a. $^{-}30$ d. $^{-}105$
   b. $^{-}30$ e. 42
   c. $^{-}105$ f. 42

5. The associative property of multiplication.

The Product of Two Negative Numbers

This concept, in contrast to the products in the last section, is usually difficult for pupils. The section is divided into several sub-sections, each of which attempts to provide a reasonable basis for the fact that the product of two negative numbers is positive.

The product of two negative numbers is first approached from multiplication tables. By showing only a small section, we set the stage for enlarging the table. All students should know the part that is filled in. Use of Exercises 17-5c is made to fill in the products of positive and negative numbers. Finally the number patterns developed in the table are extended to the missing part. The teacher will need to work with students if they are to follow this development. The class discussion that follows helps with this idea. It would be most unusual if a large number in the class accept this idea at this stage. The initial presentation is followed by a set of situation problems designed to show that a positive product is a reasonable result of multiplying two negative numbers. This should be developed orally also. Finally, an adaptation of a mathematical explanation is used to strengthen the concept.

Teaching suggestions: Read with the class the explanation of the table. If your students are not familiar with this type of table, be sure they understand how the table works. After the explanation is read, use Class Discussion 17-5c. Give the students time to do Problem 1, then let them look for patterns. If necessary, the students can be helped by copying one column and/or one row from the table as shown here.

$^{-}12$	$^{-}8$	$^{-}4$	0	4	8	12

$^{-}9$
$^{-}6$
$^{-}3$
0
3
6
9

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From the discussion, students should find that numbers decrease by a constant amount as one moves from right to left or bottom to top. This will probably be stated in reverse order. Since the extension is to the left and up, try to have the students verbalize the relationships when moving toward the blank corner. Pupils may think it reasonable that a pattern exists, but they find it hard to believe that the product of two negative numbers is a positive number. However, after other approaches are taken, the fact will seem more convincing.

The last exercise in Chapter 19 provides a graphical method of multiplying numbers. This method provides another convincing demonstration that the product of two negative numbers must be positive.

Answers to Exercises 17-5c

1. See completed chart in text.
2. The difference between any two adjacent numbers in any row is the same as the factor at the left of the row. The difference between two adjacent numbers in any column is the same as the factor at the top of the column. For the right column: each number is 4 more than the number above it. For the bottom row: each number is 4 more than the number on its left.
3. 4 3 2 1
4. The column under $^{-}1$.
5. Successive numbers decrease by 2, 3, or 4.
6. See completed chart in text.
7. 6
8. Positive.

Answers to Exercises 17-5d

1. a. The product of two positive numbers is a positive number.

   b. The product of a negative number and a positive number is a negative number.

   c. The product of two negative numbers is a positive number.

2. a. 0
   b. $^{-}8$
   c. $^{-}20$
   d. $^{-}24$
   e. $^{-}34$
   f. $^{-}245$
   g. 54
   h. 600
   i. 903
   j. 0.84
   k. $^{-}60$
   l. 576
   m. $^{-}66$
   n. $^{-}(\frac{160}{3})$ or $^{-}53\frac{1}{3}$
   o. $^{-}16$

3. a. $^{-}2$
   b. $^{-}3$
   c. $^{-}10$
   d. $^{-}4$
   e. 4
   f. $^{-}10$
   g. $^{-}1$
   h. 0
   i. $^{-}1$
   j. $^{-}6$
   k. $^{-}9$
   l. $^{-}6$
   m. 9
   n. $^{-}2$
   o. 10
   p. 2

Section 17-5e provides an opportunity to develop further the meaning of the product of two negative numbers. It needs careful development in class. Demonstrating that properties of operations observed with non-negative rational numbers hold for the negative numbers is the key to the method developed in Exercises 17-5e.

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Answers to Exercises 17-5e

1. multiplication
2. additive inverses or $2 + (^-2)$
3. distributive
4. negative number
5. 6
6. 6

Answers to Exercises 17-5f

1. a. $^-4 \cdot (3 + 8) = {^-4}(3) + (^-4)(8)$

$= {^-12} + (^-32)$

$= {^-44}$

Check: $(3 + 8) = 11$ and $^-4 \cdot 11 = {^-44}$

b. $^-2\,[(^-3) + 6] = {^-2}\,(^-3) + (^-2)(6)$

$= 6 + (^-12)$

$= {^-6}$

c. $5[4 + (^-7)] = 5\,(4) + 5\,(^-7)$

$= 20 + (^-35)$

$= {^-15}$

d. $^-10[(^-8) + (^-1)] = {^-10}(^-8) + (^-10)(^-1)$

$= 80 + 10$

$= 90$

e. $6[(^-3) + (^-9)] = 6(^-3) + 6(^-9)$

$= {^-18} + (^-54)$

$= {^-72}$

f. $^-4\,[(^-2) + 7] = {}^-4(^-2) + (^-4)(7)$

$= 8 + (^-28)$

$= {}^-20$

2. a. $n = {}^-12$
   b. $n = {}^-15$
   c. $n = {}^-5$
   d. $n = {}^-10$
   e. $n = 4$
   f. $n = 2$

3. a. 60
   b. 15
   c. 300
   d. $^-192$
   e. 192
   f. 0
   g. $^-16$
   h. 1000
   i. $^-66$
   j. $^-240$
   k. 12
   l. 42
   m. 23
   n. $^-585$
   o. 585
   p. 0
   q. $^-(\frac{3}{4})$
   r. $^-180$
   s. 60
   t. $^-420$

17-6. Division of Rational Numbers.

When students understand multiplication of positive and negative numbers, division offers no problem. Division is presented as the inverse of multiplication. If $b \cdot x = a$ then $x = \frac{a}{b}$.

Teaching suggestions: Review the relation between $b \cdot x = a$ and $x = \frac{a}{b}$ using specific numbers. Use Class Discussion 17-6a to bring out the relationship between positive and negative factors and their products.

Answers to Exercises 17-6a

1. a. $3n = 12$
   b. $3n = {}^-12$
   c. $^-3n = {}^-12$
   d. $^-3n = 12$

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2. a. a. positive
       b. negative
       c. positive
       d. negative

   b. $\frac{12}{3} = 4$, $\frac{^-12}{3} = {^-4}$, $\frac{^-12}{^-3} = 4$, $\frac{12}{^-3} = {^-4}$

3. a. positive
   b. positive
   c. negative
   d. negative

Answers to Exercises 17-6b

1. a. $^-9$
   b. $^-5$
   c. $104 \div {^-8} = {^-13}$ because $^-8 \cdot {^-13} = 104$
   d. $3 \div (\frac{^-3}{2}) = {^-2}$ because $(\frac{^-3}{2})(^-2) = 3$
   e. $^-2 \div 3 = \frac{^-2}{3}$ because $3 \cdot (\frac{^-2}{3}) = {^-2}$
   f. $^-1 \div {^-1} = 1$ because $^-1 \cdot 1 = {^-1}$
   g. $0 \div {^-3} = 0$ because $^-3 \cdot 0 = 0$

2. a. $^-28$ d. $^-72$ g. $^-3$
   b. 12 e. 72 h. 4
   c. $^-12$ f. 735 i. $\frac{2}{5}$

3. a. $^-4$ d. $^-24$ g. $^-(\frac{3}{4})$
   b. $^-4$ e. $^-8$ h. $^-10$
   c. $^-6$ f. $^-21$ i. $^-(\frac{12}{25})$

   Be sure to point out to the class the relationship between Problem 2 and Problem 3.

4. a. $\frac{4}{3}$ c. $^-1$ e. $^-(\frac{4}{3})$
   b. $\frac{^-4}{3}$ d. $\frac{4}{3}$ f. 1

5. (a) and (b)

6. $R = \{\frac{1}{6}, \frac{2}{3}, 1, \frac{6}{5}, {^-1}, {^-(\frac{4}{3})}, {^-(\frac{3}{7})}\}$

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7. a. 2
b. $^-5$
c. $^-5$
d. 5
e. $^-5$
f. 5
g. 5
h. 3
i. $^-4$
j. 13
k. 0
l. $^-25$
m. $^-36$
n. 21
o. $^-22$
p. 13

17-8. Chapter Review.

Answers to Exercises 17-8

1. a. 10
b. $^-4$
c. 0
d. $^-1$
e. 44
f. 5

2. 57¢ per doz.

3. Top parking level, 2 floors below the 1st floor.

4. a. $^-3$
b. 4
c. 4
d. 12
e. 3
f. $^-24$

5. $^-8830$ ft.

6. $^+154$ ft.

7. a. 3000
b. $^-56$
c. $^-45$
d. 63
e. $^-180$
f. 391

8. a. $^-18^{\circ}$
b. $^-6^{\circ}$

9. a. $^-5$
b. $^-4$
c. 3
d. $^-9$
e. $^-10\frac{1}{5}$ or $^-(\frac{51}{5})$
f. 203

10. $^-30$ ft. per day

17-9. Cumulative Review.

Answers to Exercises 17-9

1. a. $\frac{8}{9}$

   b. $\frac{4}{9}$

   c. $\frac{4}{27}$

   d. 3

   e. $\frac{4}{3}$ or $1\frac{1}{3}$

2. 1.4 and $\frac{91}{65}$

3. a. .003

   b. $0.2\overline{2}$

   c. $4.5\overline{0}$

   d. $333.\overline{3}$

4. a. 0.002888

   b. 0.02

   c. 6.37

5. a. $2\frac{1}{2}$ or 2.5

   b. 216

   c. $\frac{1}{4}$ or .25

   d. 10

   e. 2100

6. a. 36 cu. ft.

   b. 66 sq. ft.

7. $m(\angle A) = 75$

8. Area of square - Area of circle = area wasted

   $4 - \pi \cdot 1^2 \approx 0.86$ .86 sq. ft. are wasted.

   Or: $4 - \frac{22}{7} = \frac{6}{7}$ $\frac{6}{7}$ sq. ft. are wasted.

9. a. $\frac{n}{100} = \frac{1}{100}$, $n = 1$, 1%

   b. $\frac{n}{100} = \frac{1}{36}$, $n = \frac{100}{36} = 2.7\overline{7}$, 3%

   c. $\frac{n}{100} = \frac{5280}{1}$, $n = 528{,}000$, 528,000%

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10. a. 4

b. 0

c. $\frac{2}{5}$ or 0.4

11. a. 4

b. 7

12. a. 54

b. $^{-}54$

c. $^{-}54$

d. 54

13. a. $^{-}12$

b. $^{-}3$

14. a. 4

b. $^{-}4$

c. $^{-}4$

d. 4

d. 27

e. 260

c. $^{-}9$

d. 2

e. $^{-}48$

f. $^{-}21$

g. $^{-}10$

h. $\frac{1}{8}$

c. $^{-}6$

d. 5

e. $^{-}9$

f. $^{-}6$

g. 30

h. $^{-}3$

Sample Test Questions for Chapter 17

Note: Teachers should construct their own tests, using carefully selected items from those given here and from items of their own. Careful attention should be given to difficulty of items and time required to complete the test.

True or False: Decide whether the following statements are true or false.

(T) 1. We can associate rational numbers with points on a line.

(T) 2. The sum of positive two and negative two is zero.

(T) 3. The number zero does not have an opposite.

(T) 4. There is no greatest number on the number line.

(F) 5. All negative rational numbers are associated with points on the number line to the right of zero.

(T) 6. The point on the number line associated with 2 and the point on the number line associated with $^{-}2$ are the same distance from the point on the number line associated with 0.

(F) 7. The rational number $^{-}3$ is greater than the rational number $^{-}2$.

(F) 8. The quotient of two negative numbers is a negative number.

(T) 9. All negative numbers are smaller than zero.

(F) 10. The integers are made up of the set of counting numbers and zero.

Completion:

1. The sum of 4 and $^{-}6$ is $(^{-}2)$.
2. The point on the number line midway between 4 and $^{-}6$ is $(^{-}1)$.
3. The difference $4 - (^{-}6)$ is (10).
4. The product of 4 and $^{-}6$ is $(^{-}24)$.
5. The quotient $4 \div (^{-}6)$ is $^{-}(\frac{2}{3})$
6. If $^{-}5$ is subtracted from 5 the difference is (10).

7. The numbers which we associate with points to the left of zero on the number line are called (negative numbers) .

8. On the number line, the rational numbers assigned to two points the same distance from zero are called (opposites) .

9. The opposite of $^{-}6$ is (6) .

10. The origin is the point which corresponds to the number (zero) on the number line.

Multiple Choice: Select the letter of the answer that is correct.

(d) 1. The sum of $^{-}3$ and $^{-}6$ is
a. 3 b. $^{-}3$ c. 9 d. $^{-}9$ e. none of these

(a) 2. The difference $^{-}3 - (^{-}6)$ is
a. 3 b. $^{-}3$ c. 9 d. $^{-}9$ e. none of these

(c) 3. The product of $^{-}3$ and $^{-}6$ is
a. 9 b. $^{-}9$ c. 18 d. $^{-}18$ e. none of these

(b) 4. The quotient $(^{-}3) \div (^{-}6)$ equals
a. 2 b. $\frac{1}{2}$ c. $^{-}2$ d. $(^{-}\frac{1}{2})$ e. none of these

(d) 5. The number that you multiply $^{-}3$ by to get 1 is
a. 3 b. $^{-}3$ c. $\frac{1}{3}$ d. $^{-}(\frac{1}{3})$ e. none of these

(a) 6. $4 \cdot (1 - 2)$ equals
a. $^{-}4$ b. 8 c. $^{-}12$ d. 4 e. none of these

(d) 7. The difference 3 - 2 is the same as:
a. $3 - (^{-}2)$
b. $2 - 3$
c. $2 + (^{-}3)$
d. $3 + (^{-}2)$
e. $(^{-}2) + (^{-}3)$

Choose the missing number to make a true statement.

(c) 8. $4 + {}^{-}5 + (\quad) = 0$
a. 9 b. $^{-}9$ c. 1 d. $^{-}1$ e. none of these

(a) 9. $^{-}8 \cdot (\quad) = {}^{-}16$
a. 2 b. $^{-}2$ c. $\frac{1}{2}$ d. $^{-}\frac{1}{2}$ e. none of these

(b) 10. $^{-}15 - (\quad) = 8$
a. 7 b. $^{-}23$ c. 23 d. $^{-}7$ e. none of these

Chapter 18

EQUATIONS AND INEQUALITIES

Introduction

This chapter on equations and inequalities was designed to achieve the following objectives:

1. To give the student an intuitive feeling for solving equations and inequalities. At no time is there an attempt to formalize the solution of mathematical sentences. It is the feeling of the authors that the formal solution of sentences can best be done in an algebra class.
2. To give the student the opportunity to learn how to translate words into mathematical symbols and also to translate mathematical sentences into word sentences.
3. To build a vocabulary which will be meaningful and useful in later courses in mathematics.
4. To help the student realize that the number line, with which he is so familiar, is a useful device to display the solution sets of equations and inequalities.
5. To introduce the concept that a formula is merely a type of equation and to give the student practice in applying a formula to a particular situation.

It is estimated that this chapter will take from 12 to 15 days.

18-1. Sentences and Phrases.

The objective of this section is to teach the meaning of algebraic expressions, or "phrases." The term "phrase" is used here to agree with terminology used later in SMSG texts. The teacher may wish to use additional examples to insure that the pupils understand that a phrase represents a number.

Sufficient exercises have been provided to give the student practice in translating word phrases into symbolic language, and in translating symbols into words. The authors have purposely avoided the use of such words as "increased by", "decreased by", "more than", "less than", and "difference" in an attempt to minimize difficulties brought about by words such as these. The individual teacher, however, may wish to introduce some or all of these words during various class discussions.

The use of negative numbers throughout this section and in the remainder of the chapter will allow the student to maintain and strengthen concepts which he learned in the previous chapter.

Answers to Exercises 18-1a

1. a. 13
   b. $^{-}3$
   c. 12
   d. 0
   e. $3\frac{1}{3}$ or $\frac{10}{3}$
   f. $^{-}7$
   g. 6.5
   h. 1
   i. 17
   j. 11
   k. $1\frac{1}{2}$ or $\frac{3}{2}$
   l. 4
   m. 4
   n. $^{-}1$
   o. $^{-}3$

2. a. $x + 5$ or $5 + x$
   b. $8 \cdot x$ or $8x$
   c. $y + 7$ or $7 + y$
   d. $12 + z$ or $z + 12$
   e. $n + 6$ or $6 + n$ (The student may choose any letter he wishes)
   f. $n - 6$
   g. $\frac{n}{5}$
   h. $x - 4$

3. The translations listed below are only examples. There are many correct ways to express these ideas.
   - a. The sum of a certain number and 10.
   - b. A certain number minus 3.
   - c. Seven times a number.
   - d. A number divided by 5.
   - e. The sum of a number and negative 6.
   - f. 15 divided by a certain number.
   - g. The sum of a certain number and 19.
   - h. 17 minus a certain number.

4. a. $11, {}^{-}2, 7, \frac{1}{5}, {}^{-}5, 15, 20, 16$
   - b. $5, {}^{-}8, {}^{-}35, {}^{-}1, {}^{-}11, {}^{-}3, 14, 22$
   - c. $7, {}^{-}6, {}^{-}21, \frac{{}^{-}3}{5}, {}^{-}9, {}^{-}5, 16, 20$
   - d. $10, {}^{-}3, 0, 0, {}^{-}6,$ not possible, $19, 17$
   - e. $10\frac{1}{5}, {}^{-}2\frac{4}{5}, \frac{7}{5}, \frac{1}{25}, {}^{-}5\frac{4}{5}, 75, 19\frac{1}{5}, 16\frac{4}{5}$
   - f. ${}^{-}5, {}^{-}18, {}^{-}105, {}^{-}3, {}^{-}21, {}^{-}1, 4, 32$
   - g. $16, 3, 42, \frac{6}{5}, 0, \frac{5}{2}, 25, 11$
   - h. $22, 9, 84, \frac{12}{5}, 6, \frac{5}{4}, 31, 5$
   - i. $15, 2, 35, 1, {}^{-}1, 3, 24, 12$
   - j. $19, 6, 63, \frac{9}{5}, 3, \frac{5}{3}, 28, 8$

Answers to Exercises 18-1b

1. a. $x - 4$
   - b. $x + 7$
   - c. $30 - x$
   - d. $15x$
   - e. $\frac{x}{4}$
   - f. $x + 10$
   - g. $7x$
   - h. $x - 11$
   - i. $\frac{x}{2}$
   - j. $x - 6$
   - k. $x - 9$

2. a. 8
   - b. 19
   - c. 18
   - d. 180
   - e. 3
   - f. 22
   - g. 84
   - h. 1
   - i. 6
   - j. 6
   - k. 3

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3. The translations given are only samples. There are other correct ways to write each of these. Strive to get a variety of answers.
   a. The sum of a number and 1.
   b. 3 subtracted from some number.
   c. Twice a certain number.
   d. 15 more than a number.
   e. 18 divided by a number.
   f. A number subtracted from 20.
   g. 6 times a number.
   h. The sum of a number and negative 4.

4. a. 7
   b. 3
   c. 12
   d. 21
   e. 3
   f. 14
   g. 36
   h. 2

5. a. $^-1$
   b. $^-5$
   c. $^-4$
   d. 13
   e. $^-9$
   f. 22
   g. $^-12$
   h. $^-6$

6. a. $6 + a$
   b. $8b$
   c. $8c + 1$
   d. $8d - 3$
   e. $\frac{7e}{4}$
   f. $2f + 3$
   g. $5(g + 2)$
   h. $10 - 7h$
   i. $\frac{12}{i + 1}$
   j. $(j + 3)(j + 4)$
   k. $25 - 3k$
   l. $\frac{\ell}{2 + \ell}$

7. a. 3
   b. $^-24$
   c. $^-23$
   d. $^-27$
   e. $^-\frac{21}{4}$
   f. $^-3$
   g. $^-5$
   h. 31
   i. $^-6$
   j. 0
   k. 34
   l. 3

8. a. A number minus six.
b. Four times a number.
c. Five added to two times a number.
d. A number subtracted from five.
e. Twice a number plus fifteen.
f. Twice a number added to fifteen.
g. Nine minus three times some number.
h. A number added to twelve, then divided by two.
i. Seven times the sum of some number and one.
j. Five minus twice a number.

9. a. 15
b. $^{-}5$
c. 5
d. 7
e. 3
f. 4
g. 6
h. 1
i. 8
j. $^{-}1$
k. 16
l. 0

10. a. 6
b. 3
c. 9
d. $^{-}9$
e. 5
f. 8
g. $4\frac{1}{2}$
h. $10\frac{1}{2}$

18-2. Sentences and Their Solutions.

When the student completes this section, he should have a reasonable understanding of the difference between a phrase and a sentence. He should certainly recognize the difference between an equation and an inequality and understand what is meant by the solution set of a sentence. The student should now be able to translate simple situations into mathematical language.

To stress the significance of a solution set, the teacher may wish to introduce such examples as:

$$x = x + 3$$

or

$$0 \cdot x = 1$$

which are false no matter what number x is.

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In contrast, the sentences

$$x + 5 = 5 + x$$

or

$$7x = 7 \cdot x$$

are true no matter what number x is.

Again, the exercises are to be used only to give the student an intuitive approach to solving equations and inequalities. No attempt should be made to formalize a method to solve certain types of equations. The student should leave this section with the realization that some sentences have only one solution, and others have many solutions.

This section lends itself nicely to "I am thinking of a certain number" games in class. For instance, a student may propose the following:

> "I am thinking of a certain number. If I add 3, the result is 10. What is the number?"

This type of approach will give each youngster the opportunity of creating a problem which might puzzle the entire class. Through this technique the student may develop insight into solving more difficult equations because of a personal interest. Interest and participation may be better if the puzzles are developed spontaneously, rather than through a homework assignment.

Answers to Exercises 18-2a

1. a. $x = 2$
   b. $y = 7$
   c. $k = 2$
   d. $z = 4$
   e. $p = 6$
   f. $t = 6$
   g. $a = 25$
   h. $m = {}^{-}5$

2. a. $x = 9$
   b. $y = 10$
   c. $n = 11$
   d. $a = 10$
   e. $x = 9$
   f. $p = 14$
   g. $x = 8$
   h. $y = 8$

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3. a. $b = 3$
b. $a = 5$
c. $w = 7$
d. $d = 12$
e. $m = 6$
f. $x = {}^{-}1$
g. $y = {}^{-}8$
h. $x = 2\frac{3}{4}$

4. a. $n = 6$
b. $a = 16$
c. $k = {}^{-}16$
d. $x = 35$
e. $d = 18$
f. $h = {}^{-}15$
g. $s = 14$
h. $y = 32$

5. a. $x = 4$
b. $y = {}^{-}3$
c. $a = 18$
d. $x = 4$
e. $n = 13$
f. $x = {}^{-}32$
g. $y = 7$
h. $y = {}^{-}4$
i. $x = 2$
j. $m = {}^{-}9$
k. $m = 0$
l. $n = 0$
m. $n = 15$
n. $x = 3$
o. any number
p. $x = 0$
q. $p = 31\frac{1}{4}$
r. $y = {}^{-}4$

Answers to Exercises 18-2b

1. a. $x + 5 = 13$
b. $x - 3 = 7$
c. $8x = 24$
d. $\frac{x}{4} = 9$
e. $x + 10 = 21$
f. $7x = {}^{-}35$
g. $x - 11 = {}^{-}5$
h. $x - 6 = 15$
i. $\frac{x}{2} = {}^{-}7$
j. $2x + 6 = 4$

2. a. $x = 8$
b. $x = 10$
c. $x = 3$
d. $x = 36$
e. $x = 11$
f. $x = {}^{-}5$
g. $x = 6$
h. $x = 21$
i. $x = {}^{-}14$
j. $x = {}^{-}1$

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3. a. $x + 2 > 4$
   b. $5x < 10$
   c. $\frac{x}{7} > 2$
   d. $x - 3 > 6$
   e. $x - 5 < 13$
   f. $3x > {}^{-}9$
   g. $3x - 2 > 7$

4. a. The set of all numbers greater than 2.
   b. The set of all numbers less than 2.
   c. The set of all numbers greater than 14.
   d. The set of all numbers greater than 9.
   e. The set of all numbers less than 18.
   f. The set of all numbers greater than $^{-}3$.
   g. The set of all numbers greater than 3.

5. a. The sum of a certain number and two is five.
   b. The sum of a certain number and negative three is seven.
   c. The product of a number and 2 is equal to negative ten.
   d. If five is subtracted from a number the result is greater than nine.
   e. The product of five and a number is less than fifteen.
   f. If a number is subtracted from seven, the result is two.
   g. If three is subtracted from a number, the result is less than four.
   h. If a number is divided by 3, the quotient is greater than nine.
   i. If 7 is subtracted from a number, the result is negative two.
   j. If a number is divided by $^{-}30$, the quotient is six.

6. a. $y = 3$
   b. $z = 10$
   c. $a = {}^{-}5$
   d. The set of all numbers greater than 14
   e. The set of all numbers less than 3
   f. $k = 5$
   g. The set of all numbers less than 7

h. The set of all numbers greater than 27

i. $k = 5$

j. $c = {}^{-}180$

7. There are many such sentences. A few are listed below.

$$y + 1 = 6 \qquad y - 2 = 3$$

$$3y = 15 \qquad \frac{y}{5} = 1$$

18-3. Formulas.

The purpose of the work with formulas is to give the student a general review of formulas with which he is already familiar and to help him see how a formula is related to an equation. For those formulas that the student knows, the teacher may wish to get word translations from the class during a discussion period.

The teacher should insist that the student write each formula and show his substitutions before attempting to simplify his result. The exercises should serve as a review of evaluating phrases. There should be little stress here on memorizing the particular formulas used in the exercises.

Answers to Exercises 18-3

1. a. 22 ft.
   b. $27\frac{1}{2}$ ft.
2. a. 49 sq. in.
   b. 26 sq. in.
3. a. 225 sq. in.
   b. 54.76 sq. in.
4. 2400 sq. ft. and 2340 sq. ft. The difference is 60 square feet.
5. 21 feet
6. $\frac{625}{4}$ or $156\frac{1}{4}$ square feet

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7. a. 48 sq. in.
b. $27\frac{5}{8}$ sq. in.
c. 174.72 sq. in.

8. a. $135
b. $155
c. $378

9. a. 62.8 inches
b. 81.64 in.
c. 15.7 sq. ft.
d. 17.584 sq. ft.

10. a. 240 ft.
b. 3600 sq. ft.

11. a. 585 miles
b. 1,575 miles
c. 238 miles

12. a. 530.66 sq. units
b. 379.94 sq. units
c. 176.625 sq. units

13. The square is larger. 3.44 square feet

14. a. 108 sq. units
b. $73\frac{1}{2}$ sq. units
c. $181\frac{1}{2}$ sq. units

15. a. 11 cubic feet
b. Capacity is $82\frac{1}{2}$ gal.

16. a. 50° F
b. 212° F
c. 95° F
d. 32° F
e. $116\frac{3}{5}^\circ$ F

18-4. Graphing Solution Sets of Sentences.

After graphing solution sets on a number line, the pupil should be able to understand better the ideas involved in the meaning of "solution set". Relations of numbers should be much easier to understand when solution sets are viewed as a representation on the number line. These representations should be especially helpful in explaining solution sets of inequalities.

Since the student may tire quickly of drawing number lines for these graphs, it is suggested that the teacher prepare ditto sheets with number lines on them for class use. In fact, many of the exercises can be done in class with various youngsters writing their solutions to certain problems on the board. Colored chalk can be used effectively to emphasize the solution set on a particular graph.

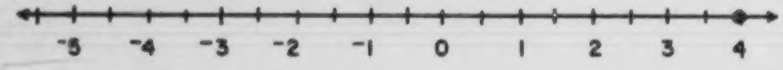
No attempt has been made to deal with compound sentences because it is felt that these are too difficult at this time. The teacher, however, may wish to use the compound sentences as extra credit problems, or as a brainbuster for some of the more ambitious students.

Answers to Exercises 18-4

1. a. $x = 4$
   b. $x = {}^-4$
   c. $x = 3$
   d. all numbers less than 1
   e. all numbers greater than 5
   f. $x = {}^-5$
   g. all numbers less than 5
   h. all numbers less than 3

page 295: 18-4

2. a.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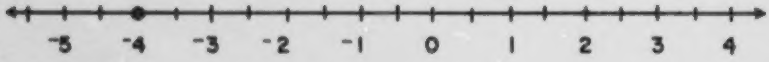
b.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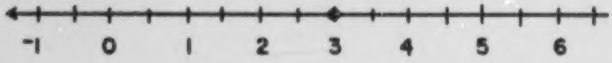
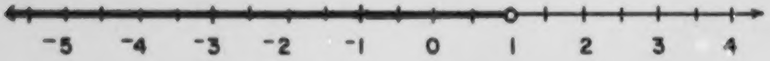
c.



d.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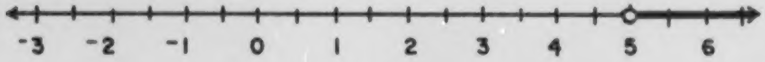
e.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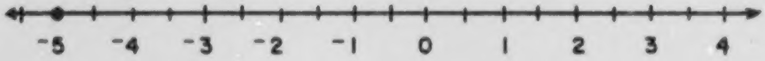
f.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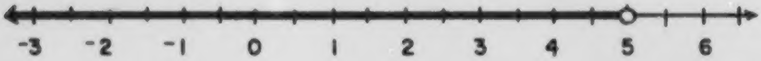
g.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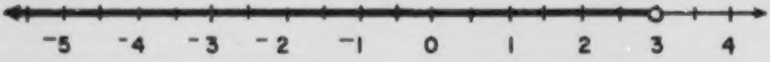
h.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4

3. a. all numbers
   b. all numbers greater than 1
   c. all numbers less than 1
   d. The empty set. (The sentence has no solution.)
   e. $w = {}^-5$
   f. $x = {}^-1$
   g. $x = {}^-1$
   *h. $x = {}^-2$

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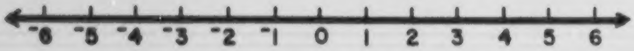
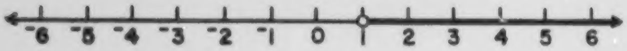
4. a.

$^-6$ $^-5$ $^-4$ $^-3$ $^-2$ $^-1$ 0 1 2 3 4 5 6

b.

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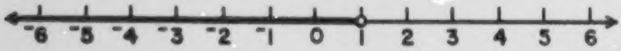
c.

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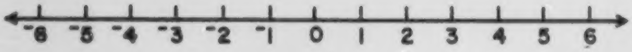
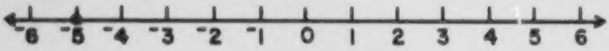
d.

$^-6$ $^-5$ $^-4$ $^-3$ $^-2$ $^-1$ 0 1 2 3 4 5 6

e.

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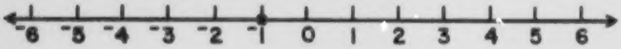
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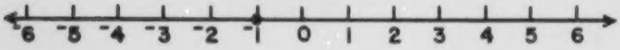
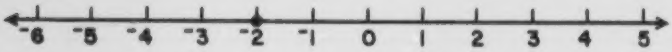
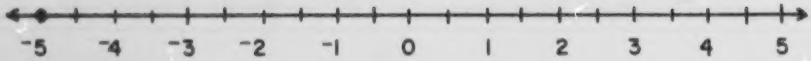
g.

$^-6$ $^-5$ $^-4$ $^-3$ $^-2$ $^-1$ 0 1 2 3 4 5 6

*h.

$^-6$ $^-5$ $^-4$ $^-3$ $^-2$ $^-1$ 0 1 2 3 4 5

5. a. $x = {}^-5$
b. $m = 2$
c. all numbers greater than 4
d. all numbers less than $^-1$
e. $m = {}^-3$
f. all numbers
g. $x = 6$
h. all numbers less than 6
i. $y = {}^-3$
j. all numbers greater than 8

6. a.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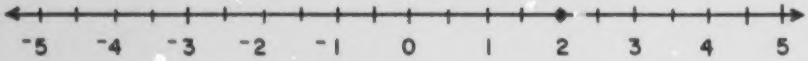
b.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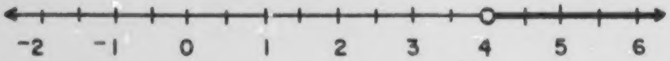

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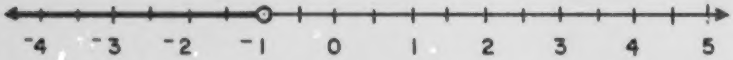
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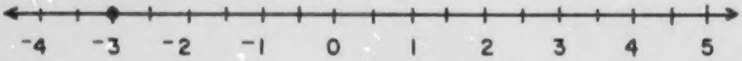
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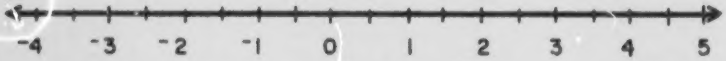
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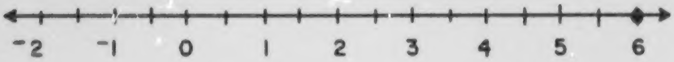
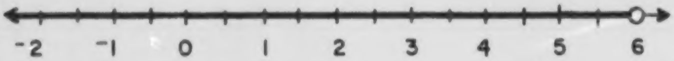
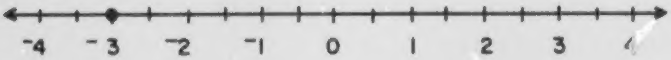


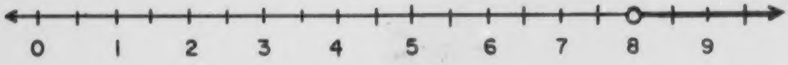
h.



i.



j.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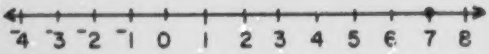
9

7. a. $7 + x = 14$

b. $x = 7$

c.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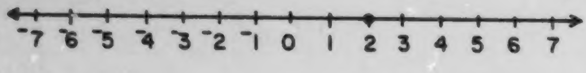
8. a. $2x + 6 = 10$

b. $x = 2$

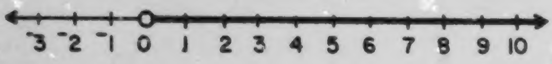
c.


-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7

9. a. $5x > 0$

b. all numbers greater than 0

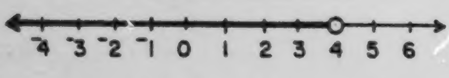
c.


-3 -2 -1 0 1 2 3 4 5 6 7 8 9 10

10. a. $x - 3 < 1$

b. all numbers less than 4

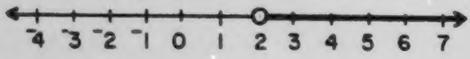
c.


-4 -3 -2 -1 0 1 2 3 4 5 6

11. a. $3x + 2 > 8$

b. all numbers greater than 2

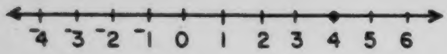
c.


-4 -3 -2 -1 0 1 2 3 4 5 6 7

12. a. $x + 2x = 12$

b. $x = 4$

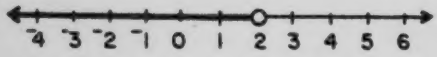
c.


-4 -3 -2 -1 0 1 2 3 4 5 6

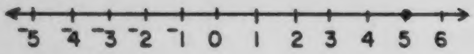
13. a. $x + x < 4$

b. all numbers less than 2

c.


-4 -3 -2 -1 0 1 2 3 4 5 6

14. a. $4x + 1 = 21$
    b. $x = 5$
    c.



18-6. Chapter Review.

Answers to Exercises 18-6

1. a. $x + 43$
   b. $11x$
   c. $3x - 6$

2. $=, <, >$

3. a. The sum of two times a number and nine.
   b. A certain number minus $^{-}2$.
   c. Twelve divided by three times a number.

4. a. 15
   b. 5
   c. $\frac{12}{9}$ or $\frac{4}{3}$ or $1\frac{1}{3}$

5. An equation will have the verb "=".

6. a. $16 + y = 12$
   b. $x + 14 > 15$
   c. $3x - 6 < 12$
   d. $4x = 20$

7. A solution set is the set of numbers which will make a sentence true.

8. a. $y = {}^-4$
   b. all numbers greater than 1
   c. all numbers less than 6
   d. $x = 5$

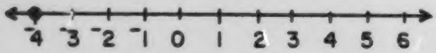
9. $y + 1 > 3$, $y - 2 > 0$ are just two possibilities

10. $16\frac{1}{3}$ feet

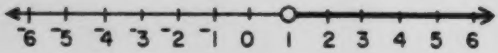
11. $\frac{625}{16}$ or $39\frac{1}{16}$ square feet

12. A graph of a sentence is a picture representation of the solution set of the sentence.

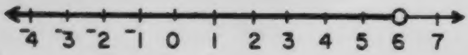
13. a.


⁻4 ⁻3 ⁻2 ⁻1 0 1 2 3 4 5 6

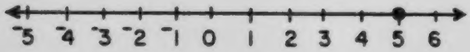
b.


⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 1 2 3 4 5 6

c.


⁻4 ⁻3 ⁻2 ⁻1 0 1 2 3 4 5 6 7

d.


⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 1 2 3 4 5 6

18-7. Cumulative Review.

Answers to Exercises 18-7

1. $\frac{{}^-8 + 10 + 0 + {}^-1 + {}^-3}{5} = \frac{{}^-2}{5}$

2. a. $\frac{29}{6}$ or $4\frac{5}{6}$
   b. 16
   c. $5\frac{7}{16}$ or $\frac{87}{16}$
   d. $\frac{7}{20}$
   e. $\frac{75}{2}$ or $37\frac{1}{2}$
   f. 6

3. 0.60, 0.6, 60%

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4. a. 1000
   b. 987.65

5. a. 13.712
   b. 0.17045
   c. 1704.5
   d. 2.1

6. 1

7. a. 0
   b. 123
   c. 320
   d. $^{-}0.8$

8. $7\frac{1}{8}$, $7\frac{3}{8}$

9. a. 8 inches
   b. 8 inches
   c. 32 inches

10. a. $\angle d$

    b. $\angle a$ or $\angle d$ or $\angle e$

    c. $\angle c$

    d. $m(\angle b) = 165$

       $m(\angle c) = 165$

       $m(\angle d) = 15$

       $m(\angle e) = 15$

Sample Test Questions for Chapter 18

Note: Teachers should construct their own tests, using carefully selected items from those given here and from items of their own. Careful attention should be given to difficulty of items and time required to complete the test.

True-False

(T) 1. If two numbers are unequal, then one must be greater than the other.

(T) 2. $4(5 - 2)$ and $\frac{6 \cdot 2}{1}$ are different names for the same number.

(T) 3. A phrase may represent one specific number.

(F) 4. The expression $x + 5$ is called a sentence.

(F) 5. $3(\frac{2}{3} + 5) < 17$

(F) 6. $\frac{4}{2} \cdot 0 = 8(\frac{1}{4})$

(T) 7. $(3x - 4)$ is a phrase.

(T) 8. "=", ">", and "<" are used as verbs in sentences.

(T) 9. $x(3 + 4) = (4 + 3)x$ is true for all values of x.

(T) 10. An inequality may be a sentence.

(F) 11. If $x = {}^-10$, then $2x + 8 = 12$.

(F) 12. If $x = 3$, then $x + 0 = 4x$.

(T) 13. If $z = 8$, then $\frac{z}{5} < \frac{z}{4}$.

Multiple Choice

1. Write a number phrase for the following.
"Multiply the difference between eight and two by three."

a. $8 - 2 \cdot (3)$
b. $(8 - 2) \cdot 3$
c. $8 \cdot 2 - 3$
d. $3(8) - 2$
e. none of these

b.

2. The solution set of the sentence $2 + x > 5$ is

a. 3
b. $^{-}3$
c. all numbers less than 3
d. all numbers greater than 3
e. 5

d.

3. The phrase $(2 \cdot 5) + 4$ represents which one of the following?

a. 8
b. 10
c. 14
d. 18
e. 40

c.

4. If x is the number of years in my age now, then my age seven years from now will be:

a. $7 + x$
b. $7 - x$
c. $7x$
d. $x - 7$
e. none of these

a.

5. If y is the number of inches in the width of a rectangle whose width is one-half its length, then its perimeter is

a. $2y(y)$
b. $2y + y$
c. $\frac{1}{2}y + y$
d. $3y$
e. $2(2y + y)$

e.

6. The area of a square whose side is s can be expressed as

a. $2s$
b. $s \cdot s$
c. $4s$
d. $2s \cdot s$
e. $s + 4$

b.

7. "A number plus four times the number is sixty," expressed in symbols is:

a. $4x = 60$
b. $x + 4 = 60$
c. $x + 4x = 60$
d. $60 - 4x = 4$
e. $x = 60(4)$

c.

8. Express the following in symbols. "Ten yards of cloth will cost more than 12 dollars" (x is the cost per yard in dollars).

a. $x > 12$

b. $x + 10 > 12$

c. $10x > 12$

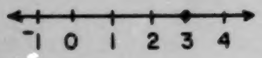
d. $\frac{10}{x} > 12$

e. $\frac{x}{10} > 12$

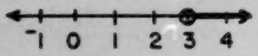
c.

9. Which of the representations on the number line below represents the solution set of the sentence $x > 3$?

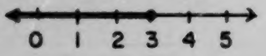
a.



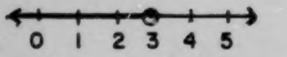
b.



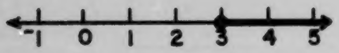
c.



d.



e.



b.

10. Which of the following is the solution of the equation $x + 7 = 3$?

a. $x = 4$

b. $x = {}^{-}4$

c. $x = 4, {}^{-}4$

d. $x = {}^{-}10$

e. none of these

b.

Completion

1. Translate each of the following into phrases using symbols.
   a. Three subtracted from the product of 30 and a number. $(30x - 3)$
   b. Five multiplied by the sum of ten and a number. $5(10 + x)$

2. Find the solution of each of the following equations.

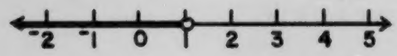
a.	$x - 8 = {}^{-}3$	(5)
b.	$3x = 5$	$(\frac{5}{3})$
c.	$10 + x = 7$	$({}^{-}3)$
d.	$\frac{x}{7} = 21$	(147)

3. Use the formula for the area of a trapezoid, $A = \frac{1}{2}h(a + b)$, and find the area of a trapezoid whose altitude is 41 feet and whose bases are 35 and 57 feet.

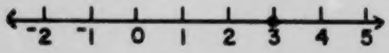
(1886 square feet)

4. Graph the solution set of each of the following sentences on the number line.

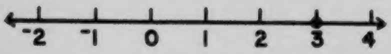
a. $2x < 2$



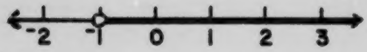
b. $x + {}^{-}2 = 1$



c. $x - 1 = 2$



d. $x > {}^{-}1$



Chapter 19

COORDINATES IN THE PLANE

This chapter is devoted to the subject of "analytic" or "coordinate" geometry of the plane. The first idea is that of employing numbers to express the position of a point in a plane. Two motivational examples are discussed in Section 1, and then the general problem is considered in Section 2. Sections 3 and 4 are devoted to graphs of equations. The only equations dealt with in the text are the equations of the form $y = bx$ whose graphs are straight lines through the origin. In the problems at the end of Section 4 there are two problems involving graphing of other types of curves, namely, the parabola $y = \frac{1}{4}x^2$ and the rectangular hyperbola $y = \frac{12}{x}$.

One observation made about the lines $y = bx$ is that they may be used to solve multiplication problems graphically. Finally, in a class discussion exercise at the end of Section 4, the line $y = (^{-}2)x$ is considered, and it is seen that, using this graphical method of multiplication, the product of two negative numbers is positive. It is hoped that this development will make it plausible for the student that the product of two negative numbers is positive.

Section 5 is devoted to the subject of distance in the plane. Here the distance between two points is found when they lie on the same horizontal line or on the same vertical line. It is seen that these methods will not suffice when the two points are otherwise situated. Therefore, it is necessary to develop more machinery in order to finish the problem of distance. This machinery is developed in Sections 6 and 7 which are devoted to the Pythagorean Theorem (Property). In Section 6 the theorem is stated and checked by measuring in particular instances. In Section 7 the theorem is proved.

In Section 8 the subject of distance is returned to and completely solved by means of the Pythagorean Theorem. Segments of length $\sqrt{65}$, $\sqrt{61}$, $\sqrt{5}$, etc., are exhibited. No hint is given that these numbers are irrational. This idea is reserved for Chapter 20. In Chapter 20 it is proved that $\sqrt{5}$ is irrational, and this fact is used as one of the cornerstones for the development of the real numbers.

There are numerous class discussion exercises in this chapter. The teacher may decide to work part of them in class and assign the rest for homework.

The suggested time allotment for this chapter is 13 days.

19-1. Locating Points in a Plane.

This section deals with two simple examples with which the student is already familiar, showing how a pair of numbers may be used to express the position of a point in a plane. The examples are: the seats in a classroom, and the street corners in a city.

Because it was felt advisable to keep the discussion short, these examples were not exploited to the utmost. Only points with non-negative integer coordinates (this word is not introduced until Section 2) were discussed. It is possible for the teacher to extend the city to the left of North Street and ask the student how coordinates could be assigned to corners so situated. In this way, coordinates such as $(^{-}3, 2)$ and $(^{-}4, 5)$ could be obtained. (East Avenue now becomes less aptly named.)

Or perhaps the teacher will want to invent a new town with Main Street and Broadway intersecting at right angles in the center of the town, thus obtaining corners in all four quadrants so as to have all possible pairs of signs, $(^{+}, ^{+})$, $(^{+}, ^{-})$, $(^{-}, ^{+})$, $(^{-}, ^{-})$, represented.

Another possible direction for generalization of the example of the city is to bring out the fact that street locations other than corners are given by number pairs in which just one of the numbers is a whole number. For example:

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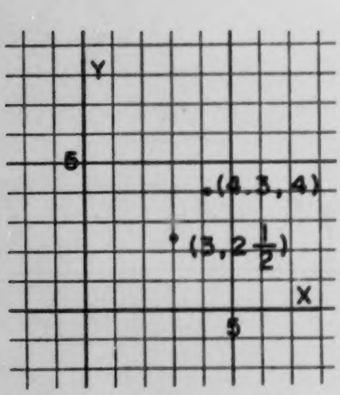

All of these extensions take time, and the teacher will have to decide for himself whether they are necessary.

Symbols of the form (a, b) are known in mathematics as ordered pairs. It should be observed that the ordered pair (2, 3) is quite different from the set {2, 3}. As we know,

$$\{2, 3\} = \{3, 2\}$$

$$\text{while} \quad (2, 3) \neq (3, 2).$$

On the other hand, we have

$$\left(\frac{3}{2}, \sqrt{4}\right) = \left(\frac{9}{6}, 2\right)$$

$$\text{since} \quad \frac{3}{2} = \frac{9}{6} \quad \text{and} \quad \sqrt{4} = 2.$$

Although the term "ordered pair" is not used in this text, it is emphasized that (3, 2) and (2, 3) denote different points.

Answers to Exercises 19-1a

1. a. Eve
   b. Nell
   c. (2, 4)
   d. (4, 2)
2. {(1, 5), (1, 2), (2, 2), (2, 1), (3, 1), (3, 3), (3, 5), (4, 3), (4, 4), (4, 5), (5, 1), (5, 4)}.

3. a. They all sit in the 4th seat of each row.
   b. {(1,4), (2,4), (3,4), (4,4), (5,4)}
   c. The second number of each pair is the same.

4. a. They all sit in the 3rd row.
   b. {(3,1), (3,2), (3,3), (3,4), (3,5)}
   c. The first number of each pair is the same.

5. a. {Gary}
   b. {(3,4)}

6. a. {John, Emma, June, May, Carl}
   b. They lie on a diagonal line that runs from the front right corner of the room to the back left corner.

7. a. {Ann, Ray, Fred, Kay, Ed, Pete, Mike, Gary, Nora, Eve}
   b. They lie above the diagonal line mentioned in Problem 6.

Answers to Exercises 19-1b

1. a. (1E, 5N)
   b. (1E, 1N)
   c. (5E, 3N)
   d. (5E, 2N)
   e. (4E, 2N)

2. a. (5E, 1N)
   b. (5E, 5N)
   c. (2E, 1N)
   d. (2E, 2N)
   e. (1E, 2N)
   f. (3E, 1N)
   g. (3E, 3N)
   h. (3E, 2N)

3. (5,1), (5,5), (2,1), (2,2), (1,2), (3,1), (3,3), (3,2)

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4. a. (0,4)
   b. (0,1)
   c. (4,0)
   d. (2,4)
   e. (2,0)
   f. (4,3)
   g. (3,3)
   h. (0,2)
   i. (1,2)
   j. (0,0)

5. a. Office Building
      Telephone Company
      Museum
      Hospital
      Zoo
   b. They lie on a diagonal line running in a Northeast direction.

6. a. City Hall
      Drug Store
      Supermarket
      Library
      Kindergarten
      Elementary School
      Jail
   b. They lie above the line mentioned in Problem 5.

7. a. Zoo, Veterinarian, Yacht Club
   b. (5,5), (5,4), (5,2)
   c. All number pairs have the same first number.

8. a. Library
      Museum
      University
      Yacht Club
   b. (1,2), (2,2), (3,2), (5,2)
   c. All pairs have the same second number.

9. a. 4
   b. 3
   c. 7

10. 5 blocks

19-2. Coordinates in the Plane.

In this section we aim to show that:

(1) every point in the plane may be represented by a number pair;

(2) every number pair represents a point in the plane.

We are on safe ground in statement (2), but in statement (1) we are cheating somewhat. We are tacitly assuming in statement (1) that for every point on the number line there is a corresponding number. Remember that so far the only numbers considered in this text are the rational numbers, and it is not true that every point on the line has a corresponding rational number. The student is not expected to notice this defect, and it should not be pointed out by the teacher at this time. In Chapter 20 the assumption is made explicitly that there is a corresponding number for each point on the number line, and that the set of all such numbers is called the set of real numbers. When this is done, statements (1) and (2) become indubitably correct with "number pair" meaning "real number pair" instead of "rational number pair."

It is possible that the student may have difficulty in making the transition from positive coordinates considered in Section 1 to the positive and negative coordinates of this section. The teacher should then point out that if we restrict ourselves to positive coordinates, we shall be able to assign coordinates only to points in a quadrant of the plane, and not to the entire plane.

It is hoped that in Exercises 19-2d, Problems 1, 2, 3, 7, and 8 will be especially helpful in showing the usefulness of coordinates in the plane. In Problem 7 the positions to be located are not exact points, and the answers should not be considered incorrect if they err by one or two tenths.

Answers to Exercises 19-2a
(Class Discussion)

Point	A	B	C	D	E	F	G	H	I	J	K	L	M
X-coordinate	3	2	$^{-}3$	2	$^{-}2$	3	$^{-}3$	$^{-}2$	7	9	$^{-}2$	$^{-}2$	10
Y-coordinate	2	3	2	$^{-}3$	3	$^{-}2$	$^{-}2$	$^{-}3$	$^{-}5$	4	$^{-}5$	2	2

Answers to Exercises 19-2b

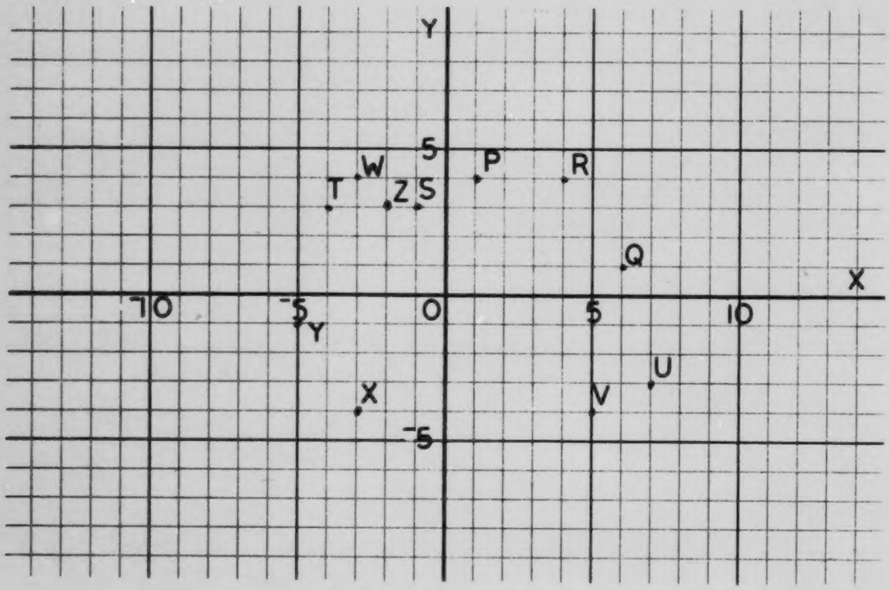
1.

Point	A	B	C	D	E	F	G	H	I	J	K	L	M
Coordinates	(3,2)	(2,3)	($^{-}3$,2)	(2,$^{-}3$)	($^{-}2$,3)	(3,$^{-}2$)	($^{-}3$,$^{-}2$)	($^{-}2$,$^{-}3$)	(7,$^{-}5$)	(9,4)	($^{-}2$,$^{-}5$)	($^{-}2$,2)	(10,2)

2. a. They lie on the same vertical line.
   b. The X-coordinates are the same.

3. a. They lie on the same horizontal line.
   b. The Y-coordinates are the same.

4.

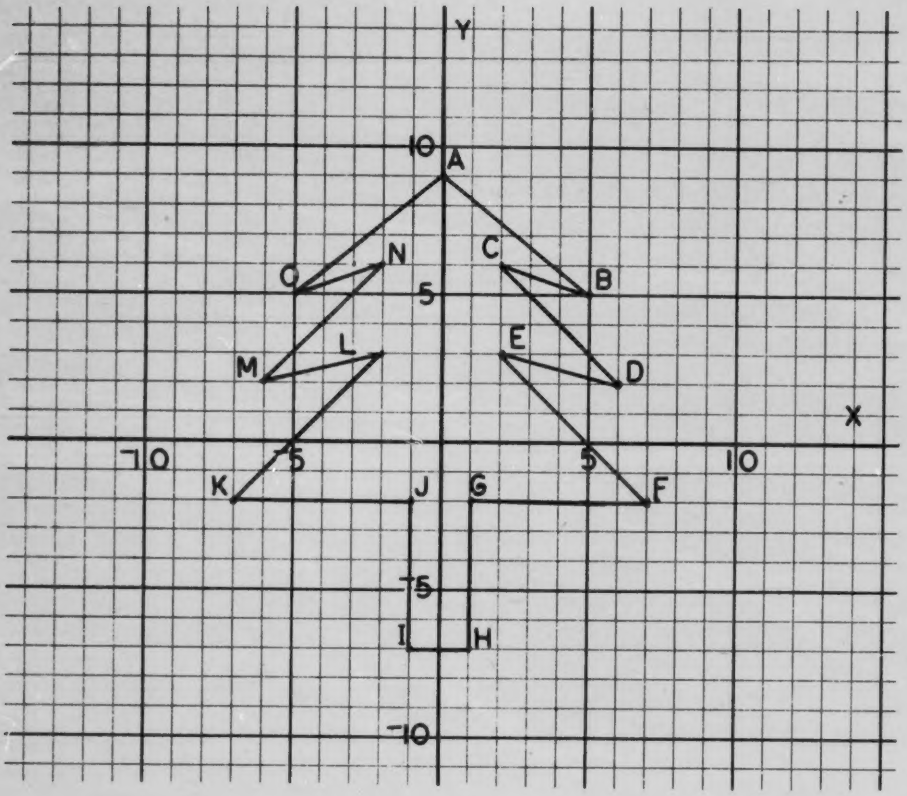


Answers to Exercises 19-2c

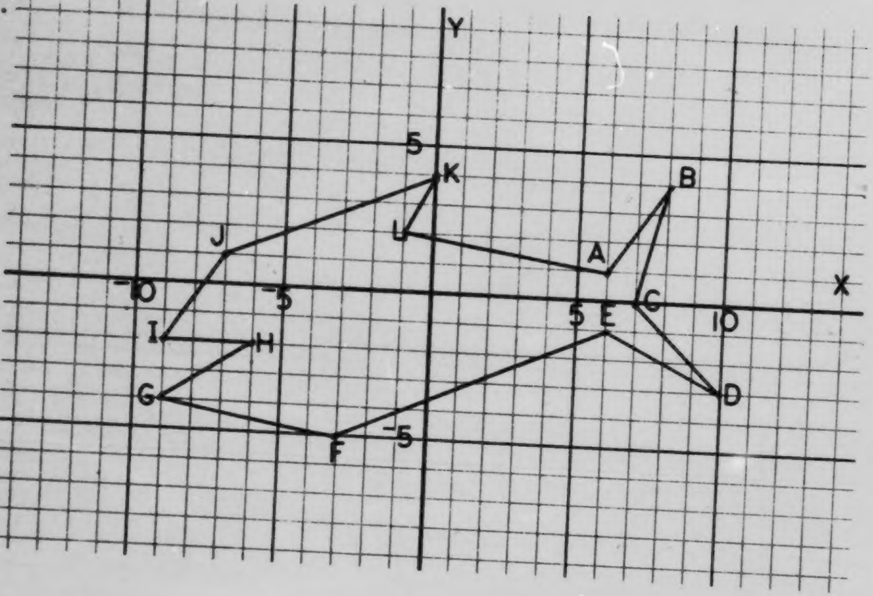
1. positive, right; positive, upper.
2. negative, left; positive, upper.
3. negative, left; negative, lower.
4. positive, right; negative, lower.

Answers to Exercises 19-2d

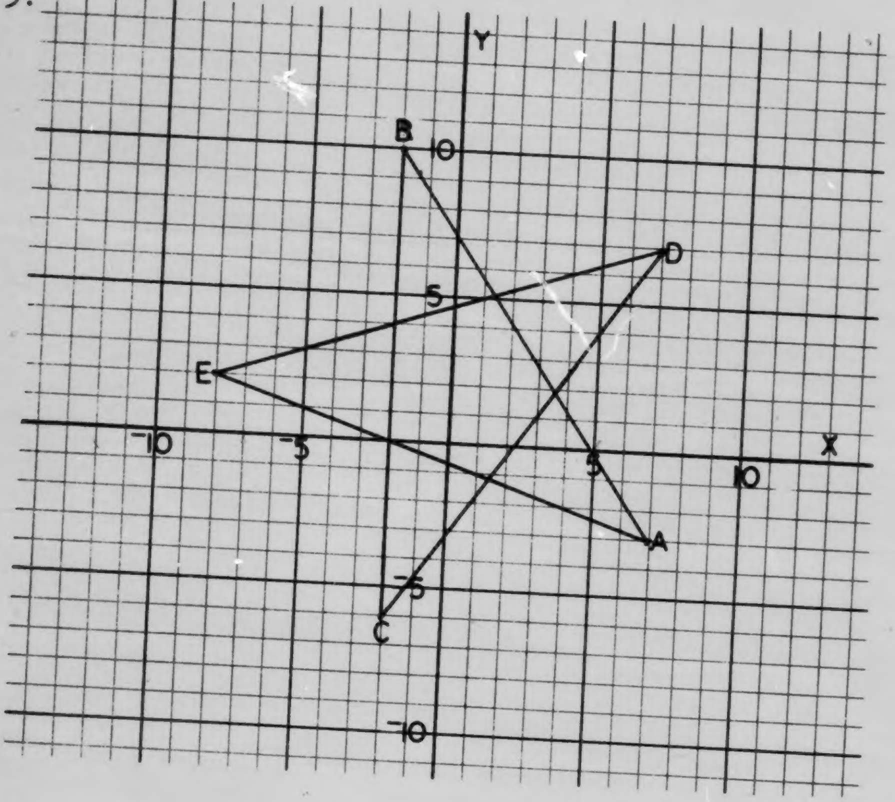
1.



2.


Y
X
5
⁻5
⁻10
⁻5
5
10
A
B
C
D
E
F
G
H
I
J
K
L

3.


Y
X
10
5
⁻5
⁻10
⁻10
⁻5
5
10
A
B
C
D
E

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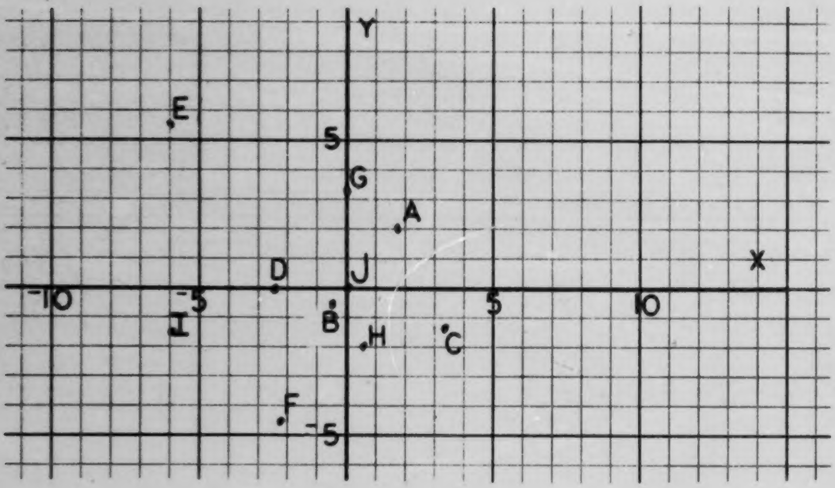
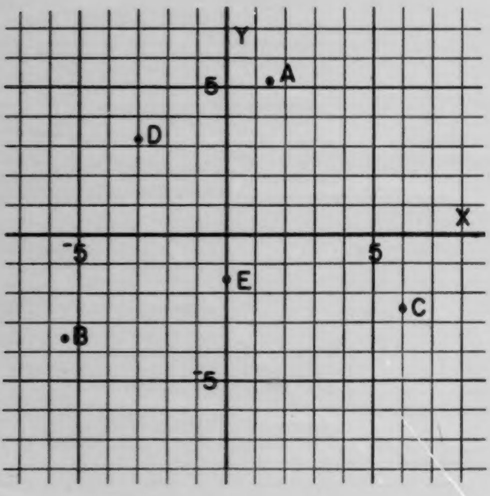
4. a. I d. III g. III j. III
b. IV e. I h. I k. II
c. III f. II i. IV l. III

5.



6.



7. a. $(0, {}^{-}4)$ e. $(0.5, {}^{-}5.5)$
b. $(7.3, 2.5)$ f. $({}^{-}3, {}^{-}5)$
c. $(1.5, {}^{-}3)$ $(3, {}^{-}1.2)$ g. $(1.5, {}^{-}3.2)$
d. $({}^{-}1, 2.5)$

(Answers to within two tenths of the above should be considered as correct.)

8. a. The island
   b. The railroad grade crossing
   c. The crossroads on the right bank of the river
   d. The wharf
   e. The railroad station
   f. The school

19-3. Graphs in the Plane.

In spite of the title, the only graphs considered are the graph of the line $y = x$ and the two half-planes determined by this line. It was felt that very little motivation could be found for setting out to find the graph of the set of those points (x, y) for which $y = x$. Would we use this graph to find the value of y when the value of x is given? This would seem pretty silly when we already know that $y = x$. Consequently, we studied the problem the other way around. We took the attitude that points in the plane provide a useful way of studying geometry. We therefore drew the line through the points $(0,0)$ and $(5,5)$ and tried to find what we could about the coordinates of points on this line.

We found that for every point (x, y) on this line we had $y = x$, and conversely, that every point (x, y) with $y = x$ lies on this line. When we say we found these facts, we do not mean that we proved them. We "found" them inductively; that is, we made the generalization from a number of examples. This is about the best that can be done without a considerable treatment of similar triangles.

Problem 2(d) in Exercises 19-3b shows an important property of this line, namely, that it bisects the angle formed by the positive X-axis and the positive Y-axis. In Problem 1 of this exercise, the students may not realize which is the greater of two negative numbers. If they do not realize, for example, that $^{-}2 > {}^{-}4$, then they will learn an important lesson when they check this graphically in Problem 2(a), (b), and (c). In Problem 3, students will have an opportunity to find, graphically, the coordinates of the point of intersection of two lines.

Answers to Exercises 19-3a

(Class Discussion)

1.

Point	A	B	C	D	E	F
X-coordinate	0	5	2	$^{-}4$	$^{-}2$	4
Y-coordinate	0	5	2	$^{-}4$	$^{-}2$	4

2. The X-coordinate is equal to the Y-coordinate.

3.

Point	G	H	J	K	L	M	N
X-coordinate	$3\frac{1}{2}$	$^{-}3\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{3}$	$^{-}\frac{3}{2}$	$^{-}\frac{1}{2}$	$^{-}\frac{9}{2}$
Y-coordinate	$3\frac{1}{2}$	$^{-}3\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{3}$	$^{-}\frac{3}{2}$	$^{-}\frac{1}{2}$	$^{-}\frac{9}{2}$

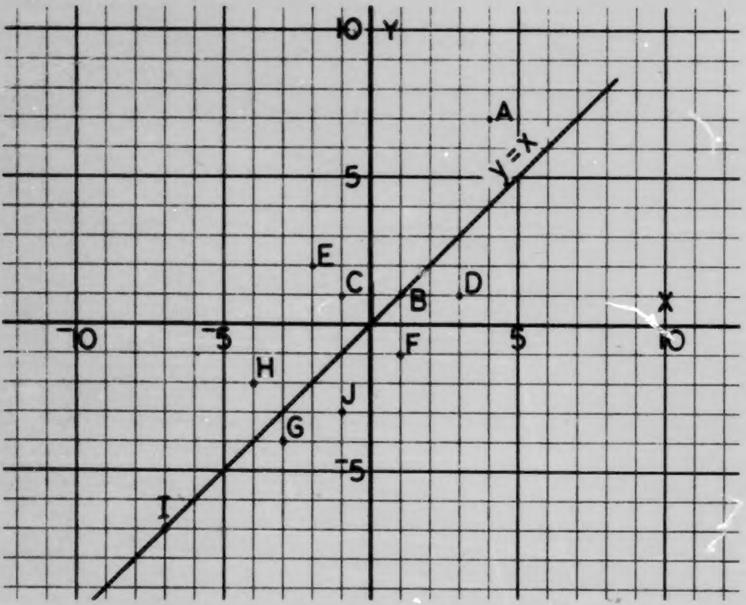
4. Yes.

5. All points on this line have equal X- and Y-coordinates.

Answers to Exercises 19-3b

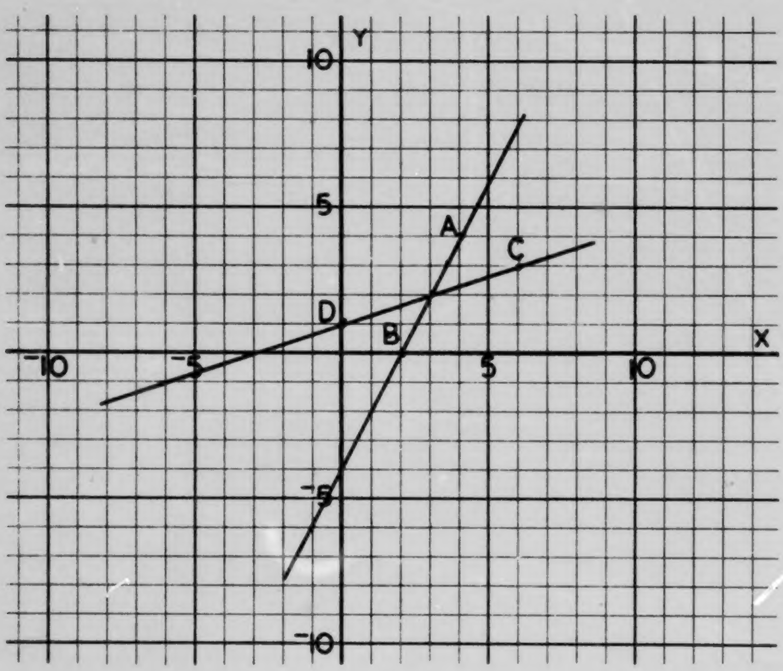
1. a. Above since $7 > 4$
   b. On since $1 = 1$
   c. Above since $1 > {}^{-}1$
   d. Below since $1 < 3$
   e. Above since $2 > {}^{-}2$
   f. Below since ${}^{-}1 < 1$
   g. Below since ${}^{-}4 < {}^{-}3$
   h. Above since ${}^{-}2 > {}^{-}4$
   i. On since ${}^{-}7 = {}^{-}7$
   j. Below since ${}^{-}3 < {}^{-}1$

2.



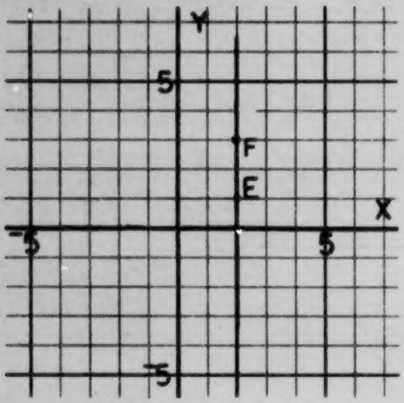
d. The measure of the angle is 45. Emphasize that the line y = x bisects the first and third quadrants.

3.



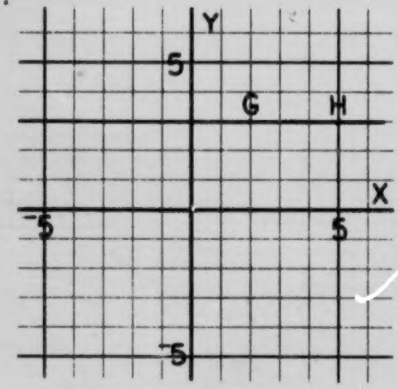
c. The point of intersection is (3,2).

4.



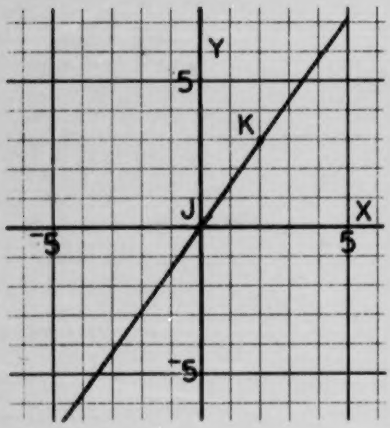
$\overleftrightarrow{EF}$ is a line parallel to Y-axis.

5.



$\overleftrightarrow{GH}$ is a line parallel to X-axis.

6.



19-4. Other Equations, Other Lines.

There are three main aims in this section.

The first aim is to show through a series of class discussion questions (Exercises 19-4a) that for various values of b, the equations

$$y = bx,$$

have graphs which are lines through the origin. The teacher will probably wish to point out that the equation $y = x$ considered in the last section is the special case in which $b = 1$.

The second aim is to observe that lines through the origin enable us to do multiplication problems graphically. To motivate, we must illustrate with an example which must satisfy these five conditions: (1) it must be an applied problem; (2) it must involve multiplying a whole list of numbers by the same factor; (3) this factor must be such as not to make the multiplication trivial; (4) the list of numbers must contain both negative and positive numbers; (5) we must be able to use the same scale on both axes and obtain a line which is neither too steep nor too flat. To meet all these conditions, it was decided to treat the problem of changing elevations in kilometers to elevations in miles.

The third aim is to fortify the conviction that the product of two negative numbers is a positive number. This is done by going on with the development of the idea that the graph of $y = bx$ gives us a geometrical way of multiplying by b. This time (in Exercises 19-4b for Class Discussion) we take $b = {}^-2$ and find that the values of y, read off the graph for the products $({}^-2)\cdot({}^-3)$, $({}^-2)\cdot({}^-1.5)$, $({}^-2)\cdot({}^-4)$, etc. are indeed positive numbers. No mention is made of the fact that we are trying to drive this point home. It is felt that the lesson will be more effective if the students notice it and comment on it themselves. If the students do not notice it themselves, then the teacher will have to take the necessary steps to see that they do.

The general problem of lines not passing through the origin is not treated in this chapter (though it would not involve a great deal of extra work to do so.) Problems 1 and 2 of Exercises 19-4c exhibit some examples of such lines, but the methods do not easily generalize to show that

$$y = bx + c$$

is the general equation of a line.

If a teacher wishes to teach the general equation of the line, the following approach, given here in skeleton form, might be used. Consider the two equations

$$(1)\quad y = 2x$$

and

$$(2)\quad y = 2x + 5.$$

For each value of x, the Y-coordinate of the point on the second curve is 5 more than the Y-coordinate of the point on the first curve. If the point on the first curve is "moved up" 5 units, we get a point on the second curve. If this is done for each point on the first curve, the effect is the same as if the whole curve $y = 2x$ is moved up 5 units. Therefore, the graph of $y = 2x + 5$ is a line parallel to the graph of $y = 2x$, and is 5 units above it.

Particularly important problems are numbers 3 and 4 of Exercises 19-4c in which the parabola $y = \frac{1}{4}x^2$ and the rectangular hyperbola $y = \frac{12}{x}$ are graphed. Here the student encounters equations whose graphs are not straight lines, and this fact should be emphasized. In Problems 5 and 7 of this set, curves are drawn from empirical data. The curve in Problem 5 is a line, and that in Problem 7 is a parabola.

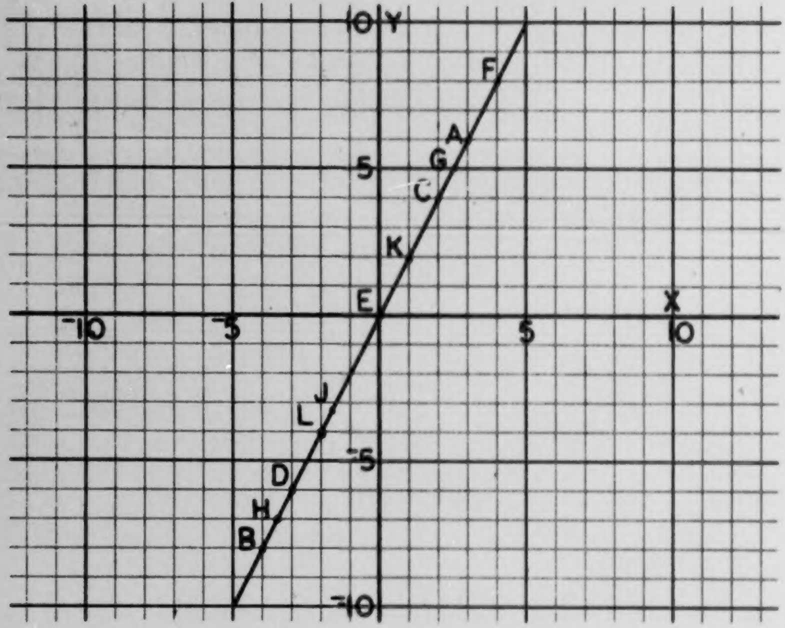
Answers to Exercises 19-4a

(Class Discussion)

1.

Point	A	B	C	D	E	F	G	H	J	K	L
X-coordinate	3	$^-4$	2	$^-3$	0	4	$2\frac{1}{2}$	$^-3\frac{1}{2}$	$^-1\frac{2}{3}$	1	$^-2$
Y-coordinate	6	$^-8$	4	$^-6$	0	8	5	$^-7$	$^-3\frac{1}{3}$	2	$^-4$

2.



3. The points lie on a line through the origin. The teacher may wish to remind the class that "curve" in mathematics includes lines and broken-lines. In Chapter 7 "curves" were thought of as special sets of points which can be represented by a pencil drawing made without lifting the pencil.

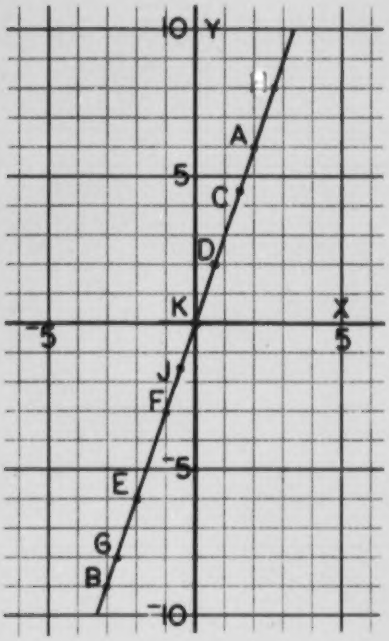
4. The Y-coordinate is $^-2$. $(^-1, ^-2)$ satisfies $y = 2x$.

5. The set of points (x, y) for which $y = 2x$ lies on the line.

page 322: 19-4

6. a.

Point	A	B	C	D	E	F	G	H	J	K
X-coordinate	2	$^{-}3$	1.5	$\frac{2}{3}$	$^{-}2$	$^{-}1$	$^{-}2\frac{2}{3}$	$2\frac{2}{3}$	$^{-}\frac{1}{2}$	0
Y-coordinate	6	$^{-}9$	4.5	2	$^{-}6$	$^{-}3$	$^{-}8$	8	$^{-}1.5$	0



b. The points lie on a line through the origin.

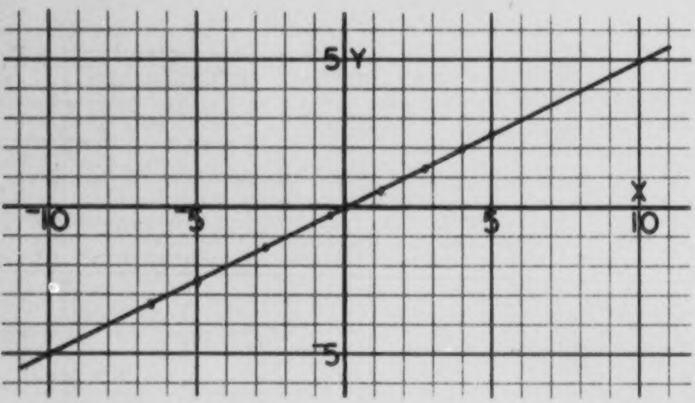
c. Yes.

page 323: 19-4

7. a.

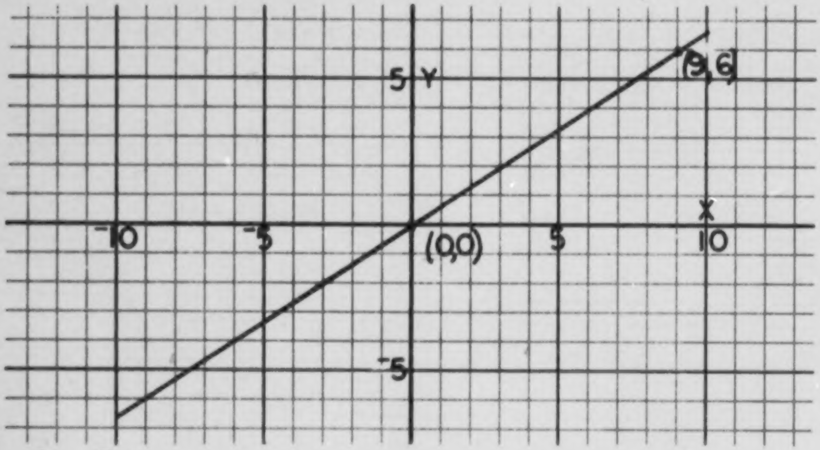
X-coordinate	0	4	$^{-}10$	7	$^{-}2\frac{2}{3}$	$^{-}5$	2.8	$^{-}6.6$	1.2	$^{-}\frac{1}{2}$
Y-coordinate	0	2	$^{-}5$	$3\frac{1}{2}$	$^{-}1\frac{1}{3}$	$^{-}2\frac{1}{2}$	1.4	$^{-}3.3$	0.6	$^{-}\frac{1}{4}$

b.



c. Yes.

8. Any points whose coordinates satisfy $y = \frac{2}{3}x$ are correct.



b. Yes.

Answers to Questions in Text

The coordinates on $y = \frac{4}{3}x$ are

$A(^{-}3, ^{-}4)$, $B(3,4)$, $C(5,6\frac{2}{3})$, $D(1\frac{1}{2},2)$.

GEOGRAPHICAL LOCATION	ELEVATION IN KILOMETERS	ELEVATION IN MILES(graph)	ELEVATION IN MILES
Matterhorn, Switzerland	4.5	2.8	2.79
Mount Everest	8.9	5.5	5.518
Dead Sea	$^{-}0.4$	$^{-}0.2$	$^{-}0.248$
Deepest Point in Pacific	$^{-}11.0$	$^{-}6.8$	$^{-}6.82$
Mount Whitney	6.2	3.8	3.844
Deepest Oil Well	$^{-}5.0$	$^{-}3.1$	$^{-}3.10$
Deepest Ocean Descent by Man	$^{-}10.9$	$^{-}6.8$	$^{-}6.758$

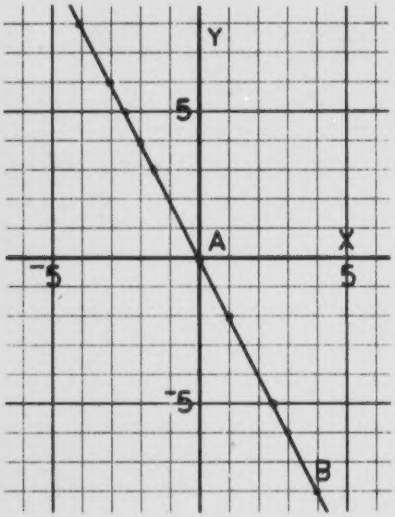
Answers to Exercises 19-4b

(Class Discussion)

1.

Point	X-coordinate	Y-coordinate
A	0	0
B	4	$^{-}8$

2. and 3.



page 327-328: 19-4

4.

x	1	3	2.5	¯3	¯1.5	3	¯2	¯2.5	¯4
(¯2)x	¯2	¯6	¯5	6	3	¯6	4	5	8

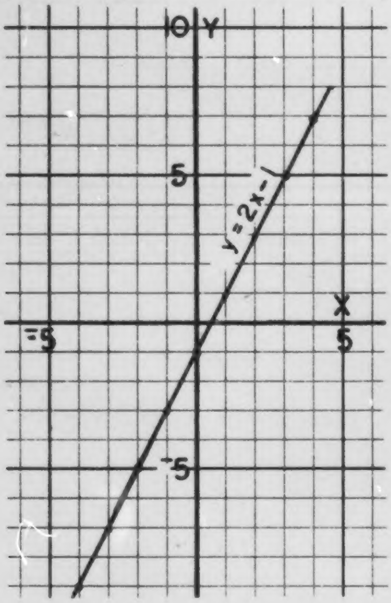
This example gives a very nice illustration that the product of two negative numbers is a positive number. The teacher should be certain to emphasize this.

Answers to Exercises 19-4c

1. a.

X-coordinate	¯4	¯3	¯2	¯1	0	1	2	3	4
Y-coordinate	¯9	¯7	¯5	¯3	¯1	1	3	5	7

b.

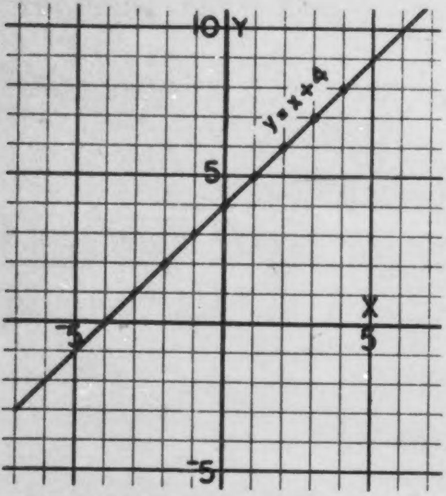


c. on a line.

page 328: 19-4

2. a.

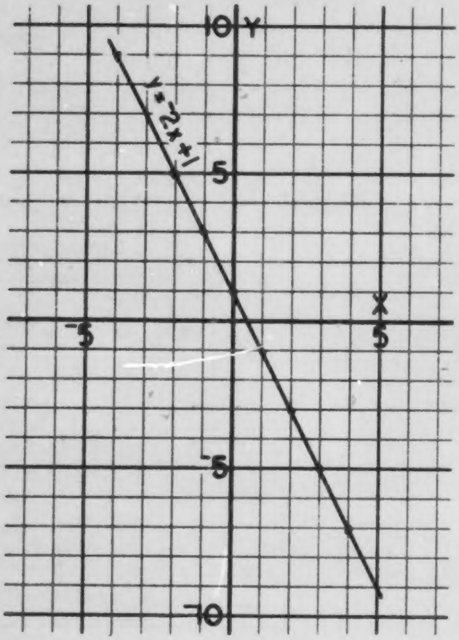
X-coordinate	$^-4$	$^-3$	$^-2$	$^-1$	0	1	2	3	4
Y-coordinate	0	1	2	3	4	5	6	7	8



This graph is a line.

b.

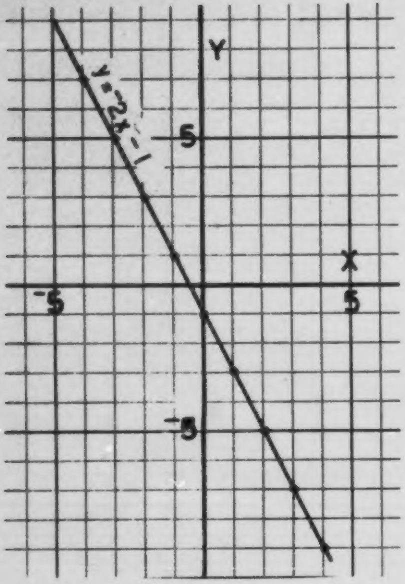
X-coordinate	$^-4$	$^-3$	$^-2$	$^-1$	0	1	2	3	4
Y-coordinate	9	7	5	3	1	$^-1$	$^-3$	$^-5$	$^-7$



This graph is a line.

c.

X-coordinate	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2	3	4
Y-coordinate	7	5	3	1	$^{-}1$	$^{-}3$	$^{-}5$	$^{-}7$	$^{-}9$

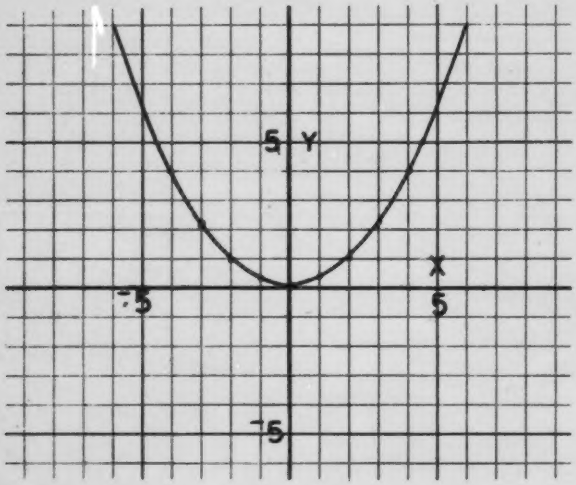


The graph is a line.

3. a.

X-coordinate	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2	3	4
Y-coordinate	4	$2\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	$2\frac{1}{4}$	4

b.

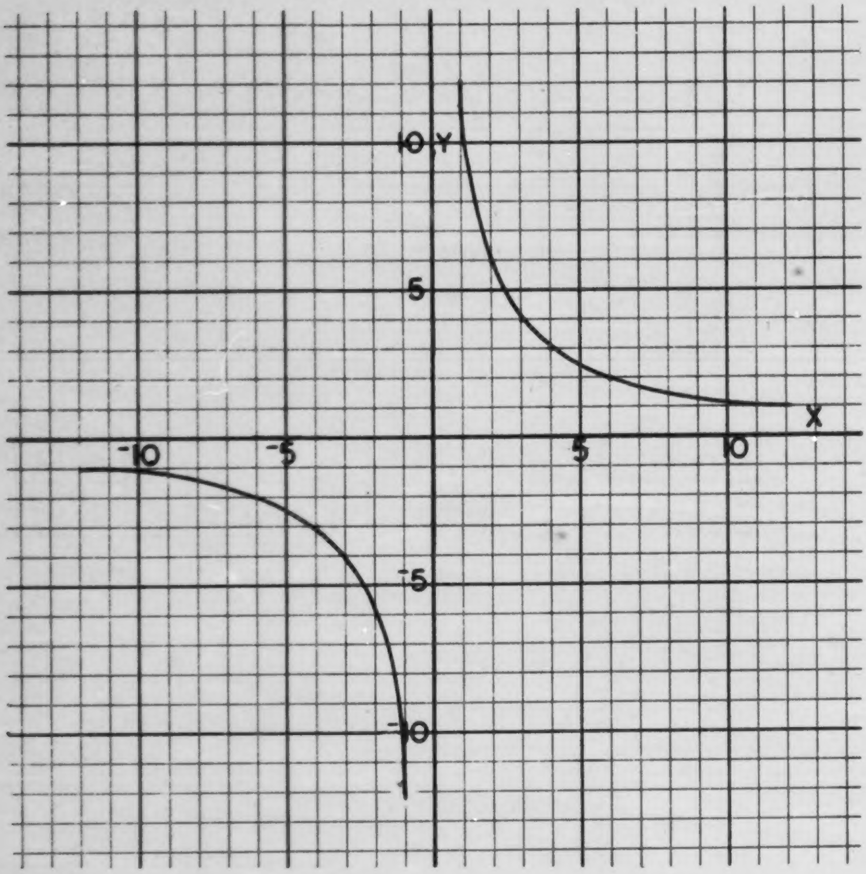


c. (This curve is called a parabola.)

4. a.

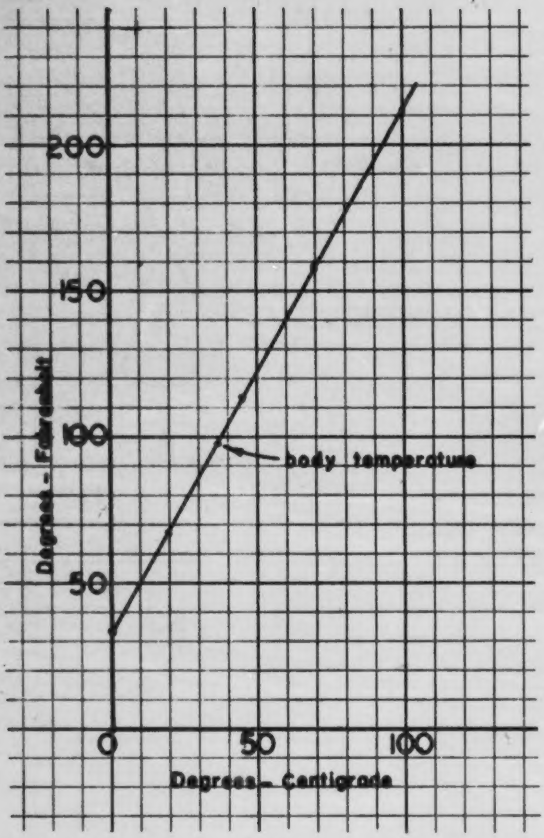
X-coordinate	$^{-}12$	$^{-}9$	$^{-}6$	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	0
Y-coordinate	$^{-}1$	$^{-}\frac{4}{3}$	$^{-}2$	$^{-}3$	$^{-}4$	$^{-}6$	$^{-}12$	none

X-coordinate	1	2	3	4	6	9	12
Y-coordinate	12	6	4	3	2	$\frac{4}{3}$	1



c. (This curve is called a hyperbola.)
Call attention to the fact that when $x = 0$, y is not defined since $\frac{12}{0}$ has no meaning. On the graph there is no point whose X-coordinate is zero.

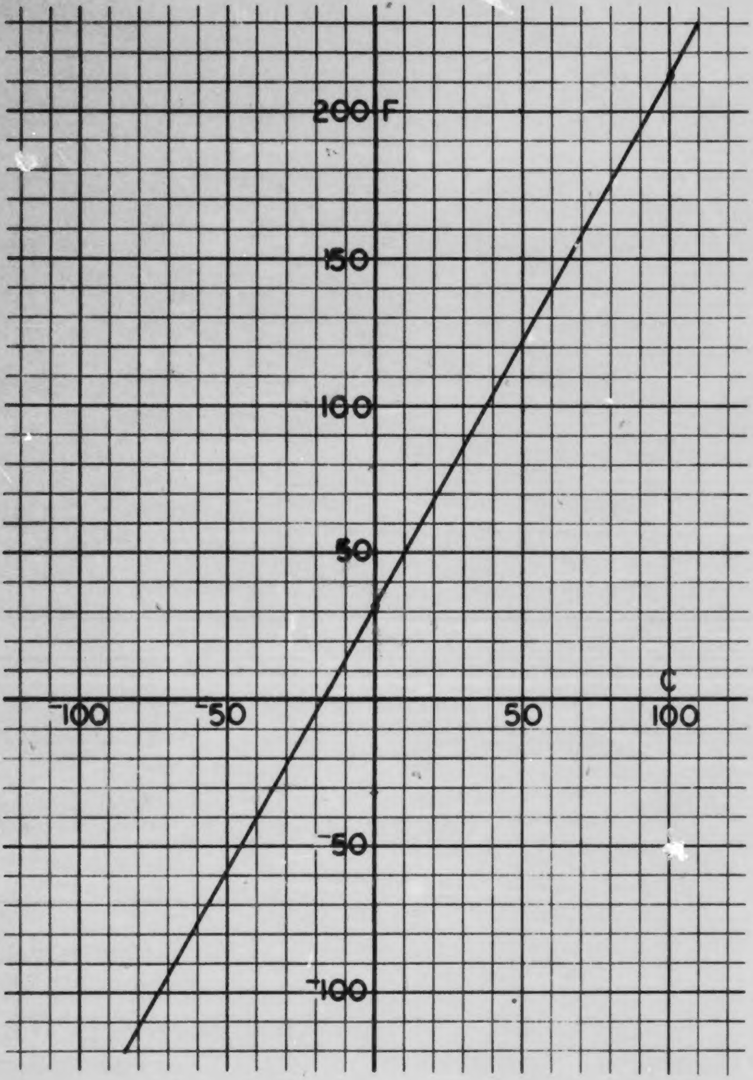
5. a. and b.



c. The points lie on a line. The teacher will realize that this can be predicted from the formula $F = \frac{9}{5}C + 32$, but students should find this linear relation after plotting the points.

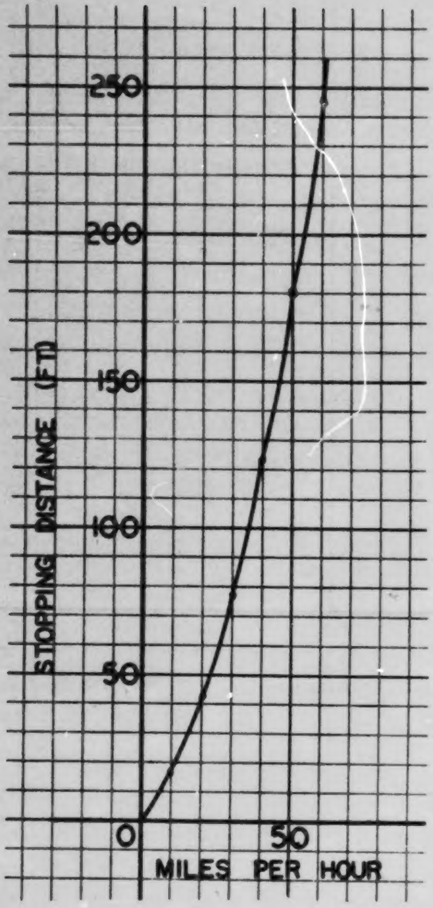
d. 98.6° F is the same as 37° C.

6.



a. No, not through quadrant IV.
b. about ⁻18° C.
c. about 58° C.
d. about ⁻68° C.

7. a.



b. The curve is not a line.

c. About 50 feet, about 110 feet, about 150 feet.

d. 53 miles per hour, 25 miles per hour.

19-5. Distance.

The principal purpose of this introduction to distance is to motivate the Pythagorean Theorem. We learn that the distance between two points on the same horizontal line is the difference of their X-coordinates, the greater minus the smaller. For example, the distance between (3,5) and (9,5) is 9 - 3 or 6. The distance between (a, b) and (c, b) is a - c or c - a, whichever of these two numbers is positive. Distance is never negative. If we had the absolute value notation at our disposal, we would say that the distance between (a, b) and (c, b) is $|a - c|$ or (which is the same thing), $|c - a|$. It is in general true that the distance between P and Q is equal to the distance between Q and P. These points are not belabored in the text, but if problems arise, the teacher will have to explain in more detail as given above.

When we come to the problem of finding the distance between two points not situated on the same vertical or horizontal line, the best we can do at this point is to construct a ruler and measure the distance. It is pointed out in detail that the unit on the ruler must agree with the unit employed in constructing the coordinate plane. A device not mentioned in the text which the teacher might use is to have the students tear off a piece of graph paper to use as a ruler.

This method of measuring is pointed out to be unsatisfactory since it will not give the exact distance when the coordinates of the points are given exactly. We need another tool with which to solve the problem of distance. This tool is the Pythagorean Theorem.

Answers to Exercises 19-5

1. a. distance 5 horizontal
   b. distance 7 vertical
   c. distance 5 horizontal
   d. distance 5 vertical
   e. distance 8 vertical
   f. distance 3 horizontal
   g. distance 15 horizontal
   h. distance 2 vertical

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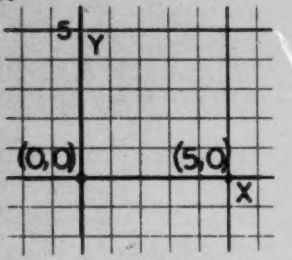
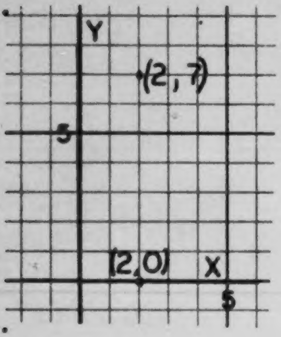
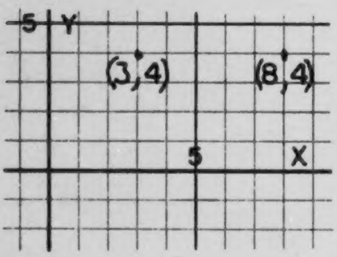
2. a.



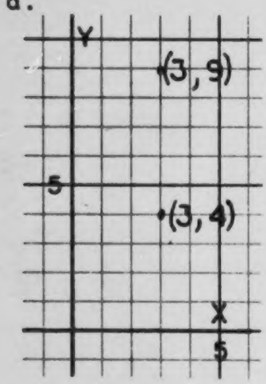
b.



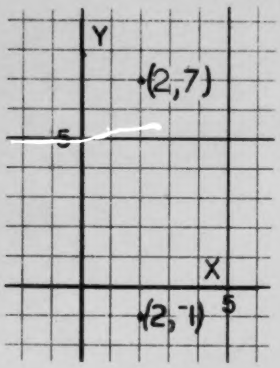
c.



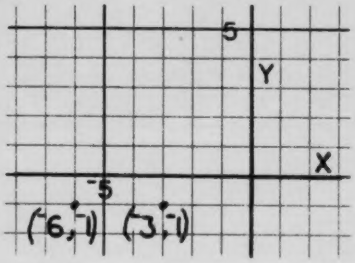
d.



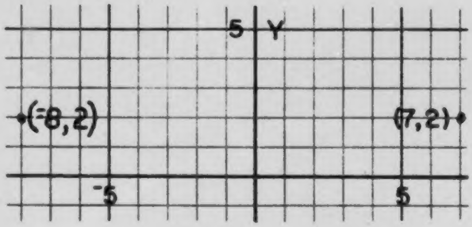
e.



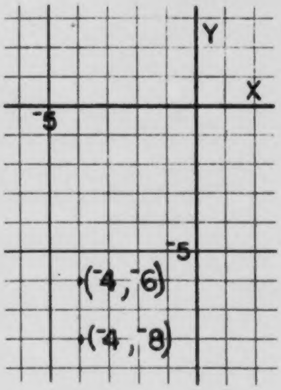
f.



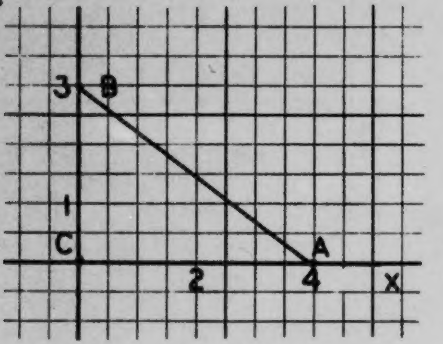
g.



h.



3. a.

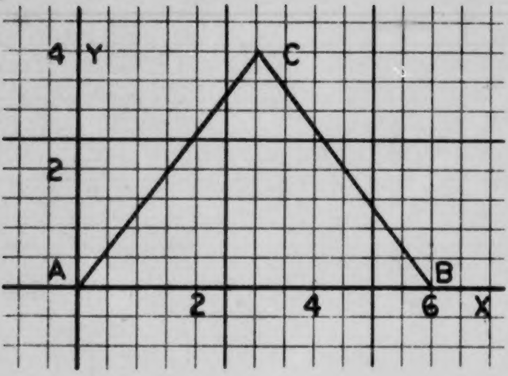


b. 4

c. 3

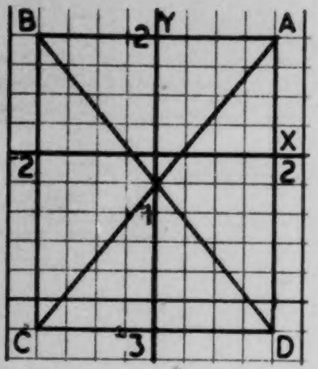
d. 5

4. a.


4
Y
C
2
A
B
2
4
6
X

b. AB = 6, BC = AC = 5

5. a.

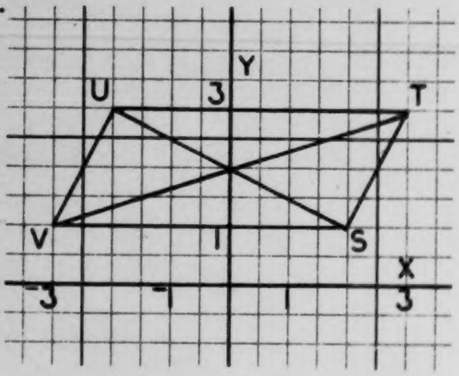

B
2
Y
A
X
‾2
2
‾1
C
‾3
D

b. AB = CD = 4 AD = BC = 5

c. a rectangle

e. $(0, {}^-\frac{1}{2})$

6. a.


Y
U
3
T
V
1
S
X
‾3
‾1
1
3

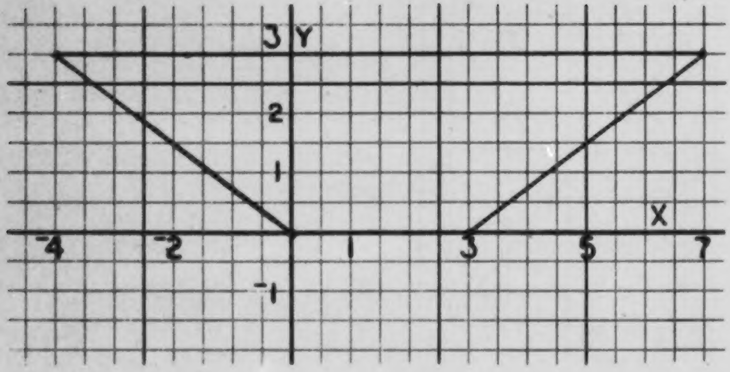
c. SV = 5

d. TU = 5

e. a parallelogram

f. (0, 2)

7.



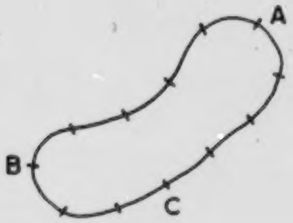
The lengths of the parallel sides are 3 and 11.

19-6. A Property of Right Triangles.

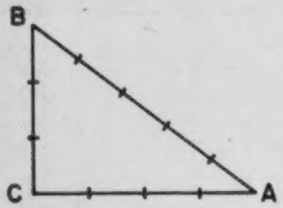
The Pythagorean Theorem (Property) is stated and verified by measurement in some particular cases. In our presentation, the theorem was not "discovered" inductively since a great deal of induction has been used in this chapter already, and furthermore, a deductive proof of the theorem is to be given in the next section.

The teacher may wish to present here the following historical material which is taken from another SMSG text.

The ancient Egyptians made use of this property of right triangles in the following way in order to make "square" corners. They took a loop of rope 12 units long.



Then they grasped it at three points which divided it into pieces 3 units, 4 units and 5 units in length. They pulled it tight to form a triangle.



Then they observed that the angle opposite the largest side is a right angle. We have already seen that a triangle with sides of length 3, 4, and 5 satisfies the condition

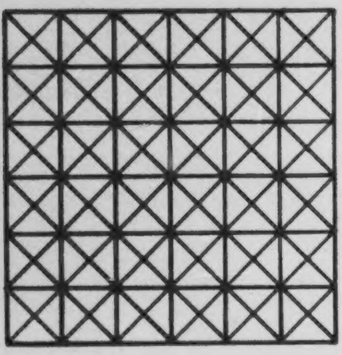
$$a^2 + b^2 = c^2$$

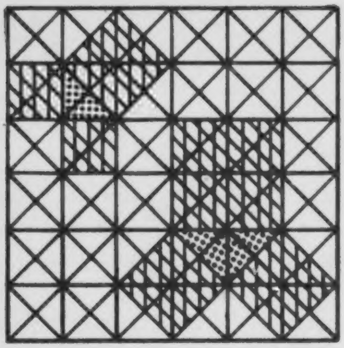
since

$$3^2 + 4^2 = 5^2.$$

Although the Egyptians made use of this property they did not prove it.

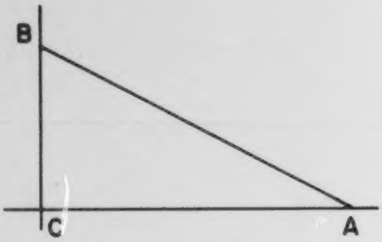
It is thought that Pythagoras looked at a mosaic like the one pictured in the first figure below. He noticed that there are many triangles of different sizes that can be found in the mosaic. But he noticed more than this. If each side of any triangle is used as one side of a square, the sum of the areas of the two smaller squares is the same as the area of the larger square. In the second figure below two triangles of different size are inked in and the squares drawn on the sides of the shaded triangles. Count the number of the smallest triangles in each square. For each triangle that is inked in, how does the number of small triangles in the two smaller squares compare with the number in the larger square? If you draw a mosaic like this, you will find that this is true not only for the two triangles given here but for a triangle of any size in this mosaic.





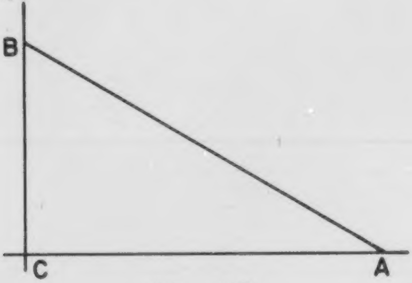
Answers to Exercises 19-6

1. a.



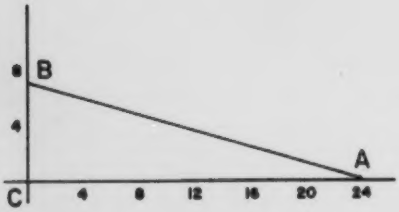
AB = 13

b.



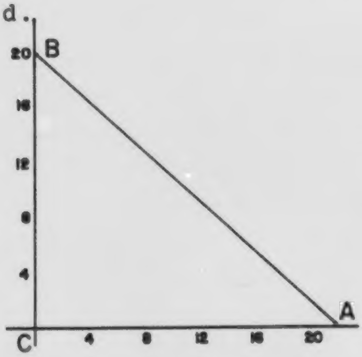
AB = 17

c.


8
B
4
A
C
4
8
12
16
20
24

AB = 25

d.


20
B
16
12
8
4
A
C
4
8
12
16
20

AB = 29

2. a. $a^2 + b^2 = 13$ $AB \approx 3.6$ $AB^2 = 12.96$

   b. $a^2 + b^2 = 41$ $AB \approx 6.4$ $AB^2 = 40.96$

   c. $a^2 + b^2 = 74$ $AB \approx 8.6$ $AB^2 = 73.96$

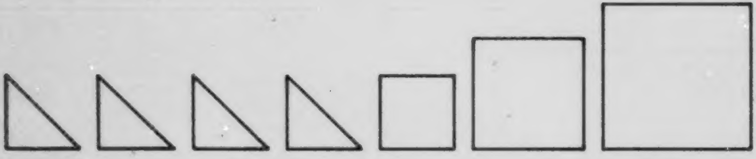
19-7. A Proof of the Pythagorean Property.

The proof of the Pythagorean Theorem given here will seem most delightful and, in fact, unforgettable to people who have seen only the more difficult and eminently forgettable traditional proofs. We hope that the proof is so nice as even to be appealing to 8th grade students encountering the theorem for the first time.

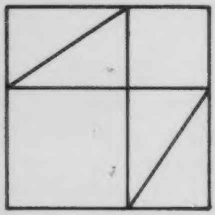
The proof will probably be more significant to the student if he actually cuts out the four copies of the right triangle and the square with edge $a + b$ and performs the coverings as indicated. The procedure can be repeated with a number of different right triangles.

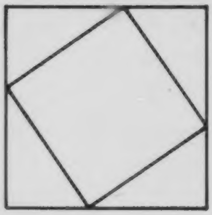
In the argument to establish that angle 3 is a right angle the statement is made that the sum of the measures of the angles at P is 180. Section 1 of Chapter 13 makes this point clear.

Another useful pedagogical device would be for the teacher to have constructed in the shop seven blocks and a box, as shown.



The students may then be shown how the blocks may be fitted into the box in the two ways.





page 342-347: 19-7,19-8

Answers to Exercises 19-7

1. a. 25
   b. 169
   c. 625
   d. 200

2. a. 144
   b. 225
   c. 441
   d. 224

3. a. 8
   b. 15
   c. 13
   d. 12
   e. 10
   f. 30
   g. 100
   h. 40
   i. 11
   j. 50

4. 90 yds.

5. 30 in.

6. 10 ft.

7. a. 13"
   b. 14 m.
   c. 80'

8. No, because the diagonal of the base is only 34 in.

19-8. Back to Distance.

Now that we have the Pythagorean Theorem, we are ready to show how to find the distance between any two points in the plane whose coordinates are given. This distance formula is expressed in the text as

$$AB^2 = (\text{difference of X-coordinates})^2 + (\text{difference of Y-coordinates})^2.$$

This formula is usually written:

Distance between (x_1,y_1) and (x_2,y_2) $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

It was felt that this notation is too difficult for the students to master.

The square root symbol is encountered for the first time in this chapter. The teacher should note that

$$\sqrt{b}$$

denotes only the positive number whose square is b. So, while

$$2^2 = 4 \quad \text{and} \quad ({}^-2)^2 = 4$$

we have

$$\sqrt{4} = 2 \quad \text{but} \quad \sqrt{4} \neq {}^-2.$$

This convention is in universal use today. In former times one frequently saw statements such as $\sqrt{4} = \pm 2$. Such statements are incorrect according to our definition. The reason for this convention is that we want our symbol to stand for a definite number and not just some indefinite member of a set of numbers. If we wish to denote the negative number whose square is 5, we must write ${}^-\sqrt{5}$.

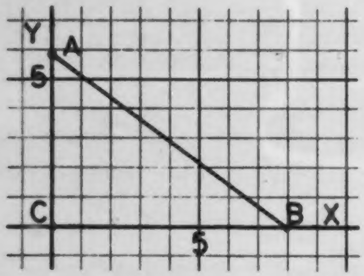
The student encounters numbers in this section such as $\sqrt{5}$, $\sqrt{61}$, $\sqrt{65}$ which are not integers. It is not even hinted that these numbers are not rational. This is the province of Chapter 20. The student is supplied with a table of approximate decimal square roots of the numbers from 1 to 100. The fact that these square roots are approximate should not lead the student to conclude that the numbers are irrational, since numbers such as $\frac{1}{7}$ cannot be expressed as terminating decimals either.

The table referred to contains squares of numbers as well as square roots, and thus lists all perfect squares between 1 and 10,000. It may be necessary to teach the pupils how to use the table, since this may be their first encounter with such an arrangement.

Answers to Exercises 19-8

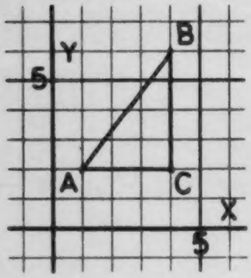
1. a.

a = 8
b = 6
c = 10



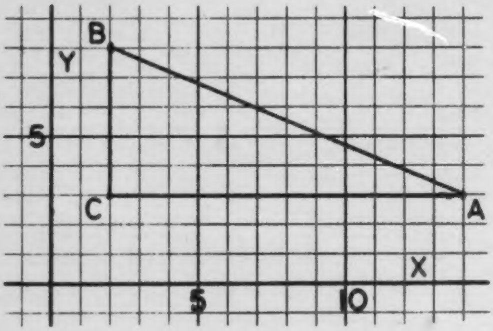
b.

a = 4
b = 3
c = 5



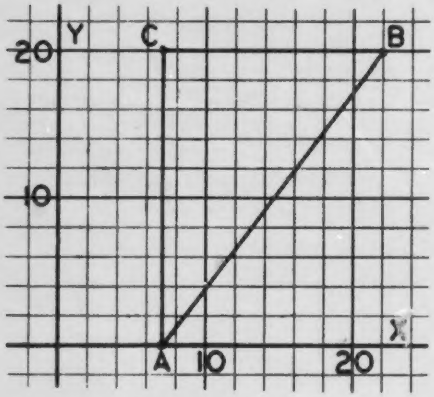
c.

a = 5
b = 12
c = 13



d.

a = 15
b = 20
c = 25


Y
C
B
20
10
X
A
10
20

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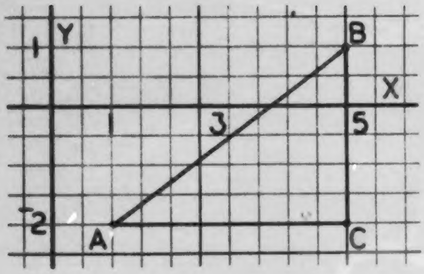
e.

$a = 3$
$b = 4$
$c = 5$



2. a. 50
   b. 29
   c. $\sqrt{13}$
   d. 9
   e. 5
   f. 25
   g. 10
   h. $\sqrt{52}$

3. a. 3
   b. 5
   c. 7
   d. 13
   e. 20
   f. 1
   g. 100
   h. 60
   i. 18
   j. 17

4. a. 4, 9, 2, 3 or $2 < \sqrt{6} < 3$
   b. 64, 81, 8, 9 or $8 < \sqrt{72} < 9$

5. a. between 2, 3
   b. 3, 4
   c. 7, 8
   d. 6, 7
   e. 8, 9
   f. 5, 6
   g. 4, 5
   h. 31, 32

6. a. 9.85
   b. 7.07
   c. 8.66
   d. 95 (exact) These perfect squares appear in the table.
   e. 64 (exact)

7. 9.8 ft.

8. a. The distance from (2,5) to $(^-3, ^-5)$ is $\sqrt{125} \approx 11$.
   b. The distance from $(0, ^-4)$ to $(7\frac{1}{2}, 2\frac{1}{2})$ is $\sqrt{98.5} \approx 10$.

9. $\sqrt{296 - 7} = 17$

19-10. Chapter Review.

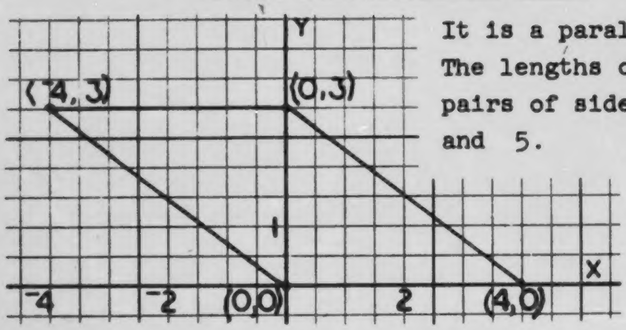
Answers to Exercises 19-10

1. horizontal
2. vertical
3. upper
4. X, negative
5. half-plane
6. line, origin
7. I, III
8. II, IV
9. 9
10. a. IV
    b. III
    c. I
    d. II
    e. IV
    f. III
11. $\sqrt{2}$
12. 41
13. a. 6.557
    b. 5.196
    c. 9
    d. 11
    e. 67
14. 5 inches

19-11. Cumulative Review.

Answers to Exercises 19-11

1.



It is a parallelogram. The lengths of opposite pairs of sides are 4, and 5.

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2. a. $\frac{7}{10}$

   b. $\frac{{}^{-}4}{21}$

   c. 2

3. $\{{}^{-}10,\ {}^{-}1.1,\ {}^{-}1.099,\ 0.001,\ 0.009,\ 0.01\}$

4. $\frac{9}{12} < \frac{35}{44}$ since $9 \times 44 < 12 \times 35$ or $396 < 420$

5. $^{-}1$

6. a. $\frac{{}^{-}7}{3}$

   b. $^{-}3$

   c. 1

7. $V = 15 \times 12 \times 9 = 1620$
   Capacity of storage tank is 1620 cu. ft.
   $V = \pi \times 3^2 \times 10 \approx 282.6$
   Capacity of truck is approximately 282.6 cu. ft.
   Since $5 < 1620 \div 282.6 < 6$, it will take 6 loads.

8. a. 55
   b. 55
   c. isosceles

9. a. $V = 8 \times \pi \times \frac{5}{2} \times \frac{5}{2} = 50\pi \approx 157$
   The volume is about 157 cu. ft.

   b. $A = 5\pi \times 8 + 2 \times \pi \times \frac{5}{2} \times \frac{5}{2} = 52\frac{1}{2}\pi \approx 164.85$
   The surface area is about 165 sq. ft.

Sample Test Questions for Chapter 19

Note: Teachers should construct their own tests, using carefully selected items from those given here and from items of their own. Careful attention should be given to difficulty of items and time required to complete the test.

True or False

(F) 1. The point whose coordinates are $(3,2)$ is the same as the point whose coordinates are $(2,3)$.

(F) 2. The point $(^-4, ^-1)$ is located in the second quadrant.

(F) 3. Each point on the line $y = 2x$ has its X-coordinate twice its Y-coordinate.

(T) 4. The graph of an equation is the set of points in the plane whose coordinates belong to the solution set of the equation.

(F) 5. A right triangle may have two right angles.

(T) 6. The length of the hypotenuse of a right triangle is less than the sum of the lengths of the other two sides of the triangle.

(T) 7. The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides.

(F) 8. The hypotenuse of a right triangle is on one of the rays that form the right angle.

(F) 9. The Y-coordinate of points lying on the Y-axis is zero.

(T) 10. All points lying in the upper half-plane (above the X-axis) have positive Y-coordinates.

(F) 11. The Pythagorean relationship is true for all triangles.

(F) 12. The length of the diagonal of a square is equal to the sum of the lengths of two of its sides.

(F) 13. The point whose coordinates are $(^-1, 1)$ lies on the line $y = x$.

(T) 14. The point whose coordinates are $({}^{-}4,{}^{-}2)$ is above the line $y = x$.

(F) 15. The point whose coordinates are $({}^{-}4,{}^{-}2)$ lies on the line $y = 2x$.

Multiple Choice

1. The points $A(2, {}^{-}3)$, $B(2,2)$, and $C(2,8)$
    - a. are the vertices of a triangle.
    - b. lie on a line parallel to the X-axis.
    - (e.) c. lie on a line that passes through the origin.
    - d. lie on a horizontal line.
    - e. none of the above is correct.

2. The distance between $A(6,0)$ and $B({}^{-}4,0)$ is
    - a. 0
    - b. 2
    - (c.) c. 10
    - d. 24
    - e. none of these

3. The graph of the inequality $y > x$
    - a. is a line through the origin.
    - b. contains every point whose Y-coordinate is equal to its X-coordinate.
    - (c.) c. is a half-plane lying above the line $y = x$.
    - d. is a half-plane lying below the line $y = x$.
    - e. is none of the above.

4. The point $(3,{}^{-}1)$ lies
    - a. in Quadrant II
    - b. in Quadrant III
    - (c.) c. in Quadrant IV
    - d. on the X-axis
    - e. none of the above is correct

5. Which of the following is a right triangle, if a, b, and c represent the lengths of the sides?

a. $a = 1$, $b = 2$, $c = 3$
b. $a = 2$, $b = 3$, $c = 4$
(c.) c. $a = 3$, $b = 4$, $c = 5$
d. $a = 4$, $b = 5$, $c = 6$
e. $a = 5$, $b = 6$, $c = 7$

6. Given the two numbers 5 and 12, the sum of their squares is

a. 13
b. 17
(e.) c. 25
d. 289
e. 169

7. Which one of the following statements is true?

a. The coordinates of a point which lies in Quadrant IV are both negative.
b. The graph of $y = 3x$ is a line which contains the point $({}^-3, {}^-1)$.
(c.) c. If you know the length of the diagonal of a square you can find the length of the side of the square.
d. The X-coordinate of points lying on the X-axis is zero.
e. The graph of the equation $y = \frac{1}{4}x^2$ is a line.

Completion

1. The Y-coordinate of the point $(3, {}^-7)$ is ___(¯7)___.

2. $({}^-4, {}^-3)$ names a point located in the ___(III)___ quadrant.

3. Each point in the coordinate plane has ___(2)___ numbers associated with it.

4. a. Find c^2 when $c^2 = a^2 + b^2$ and $a = 6$ and $b = 9$. (117)

   b. How can you represent the value of c in part (a)? $(\sqrt{117})$

5. The graph of $y = 2x$ is a (line) .

6. The point of intersection of $y = x$, $y = 2x$, $y = 3x$ is (the origin) . [Note: (0,0) is also correct.]

7. The distance between $(3, {}^{-}4)$ and $({}^{-}7, {}^{-}4)$ is (10) .

8. The hypotenuse of a right triangle is 25 units in length, and one side is 24 units in length. The third side is (7) units in length.

9. State the Pythagorean Property in words.

10. The graph of $y > x$ is (a half-plane) .

11. Find the distance between the points whose coordinates are (1,3) and (6,15). (13)

Chapter 20

REAL NUMBERS

A serious study of the real numbers is far beyond the capabilities of eighth graders. On the other hand, it is well to warn the students that not all the numbers that they work with, such as $\sqrt{2}$ and π, are rational. That is about all that this chapter attempts to do.

After establishing the agreement in Section 20-1 that every point on the number line corresponds to a number, the student is shown how to construct, by use of the Pythagorean Theorem, the number $\sqrt{2}$ in Section 20-2. Then in Section 20-3 this number $\sqrt{2}$ is shown to be irrational. The material on infinite decimals in the starred Section 20-4 is somewhat more difficult. It is left to the discretion of the teacher whether this material should be covered.

The treatment of the real number system in the following references will provide much worthwhile supplementary material:

1. Bell, E. T. MEN OF MATHEMATICS. New York. Simon and Schuster, 1937.
   Life of Georg Cantor, Chap. 29, pp. 555-579.
2. College Entrance Examination Board; Report to the Commission on Mathematics, Appendices: College Entrance Examination Board, c/o Educational Testing Service, Box 592, Princeton, N.J. Part 1, Algebra. See especially pp. 28-35, a classroom approach to Irrational Numbers.
3. Courant, Richard and Robbins, Herbert. WHAT IS MATHEMATICS? New York. Oxford University Press, 1953. Chapters 1 and 2.
4. Gamow, George. ONE, TWO, THREE . . . INFINITY. New York. Viking Press, 1947. Chapter 1, Big Numbers. See especially pp. 14-23.
5. Kasner, E., and Newman, J. MATHEMATICS AND THE IMAGINATION. New York. Simon and Schuster, 1943.

6. National Council of Teachers of Mathematics, Twenty-third Yearbook: INSIGHTS INTO MODERN MATHEMATICS. Washington, D.C., 1957. Chapter II, The Concept of Number.

7. National Council of Teachers of Mathematics, Twenty-fourth Yearbook: THE GROWTH OF MATHEMATICAL IDEAS, Grades K-12; Washington, D.C., 1959. Chapter 2, Number and Operation; Chapter 11, Promoting the Continuous Growth of Mathematical Concepts.

8. Niven, Ivan. NUMBERS: RATIONAL AND IRRATIONAL. New York. Random House, Inc., 1961.

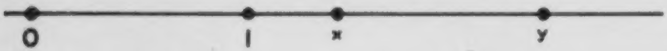
20-1. Real Numbers as Points on the Number Line.

The purpose of this section is to come out boldly with the statement: every point on the number line has a corresponding number. It is felt by the authors that the students may not have such a strong feeling for the plausibility of the statement as for the equivalent form: every segment has an exact length. The section is therefore devoted to showing the equivalence of these statements.

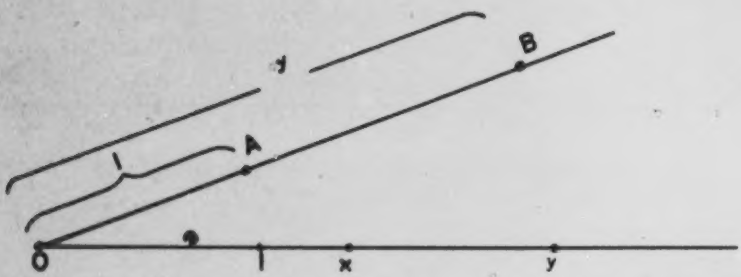
The set of numbers corresponding to points on the number line is called the set of real numbers. It will be shown in Section 20-3 that not all real numbers are rational.

Addition and subtraction may be performed by the geometrical method introduced in earlier chapters. There is also a geometrical method of multiplication which was not included in the student text.

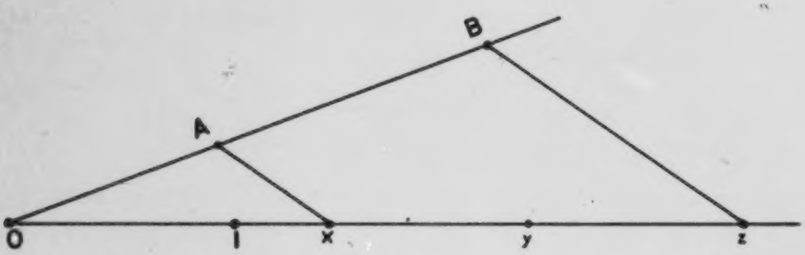
We will show how to multiply x and y.



Step I: Draw any line through the origin and mark off on it points A and B at distances 1 and y from the origin.



Step II: Draw the line joining A and x. Then, through B draw a line parallel to this last line. This parallel through B intersects the number line at a number z .



Now, from similar triangles we see that $\frac{z}{x} = \frac{OB}{OA}$ or $\frac{z}{x} = \frac{y}{1}$ or $z = xy$.

It is worthy of note that in order to add numbers on the number line, we need to know the location of 0, but in order to multiply on the number line, the location of both 0 and 1 must be known. With 0, x and y located as shown, if the location of 1 is changed, then the location of xy will also be changed.

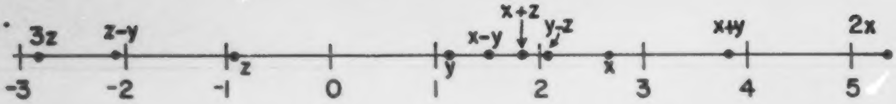
A similar construction may be made for division using the proportion

$$\frac{w}{1} = \frac{x}{y} .$$

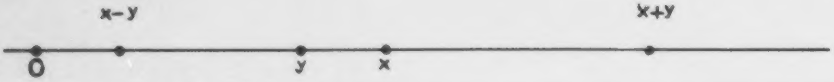
The operations of addition and multiplication of real numbers defined in this geometrical way can be shown to possess the closure, commutative, associative, inverse, distributive properties.

Answers to Exercises 20-1

1.



2.



As long as zero is marked on the number line, even if one is not marked, we can add and subtract. Multiplication is not possible.

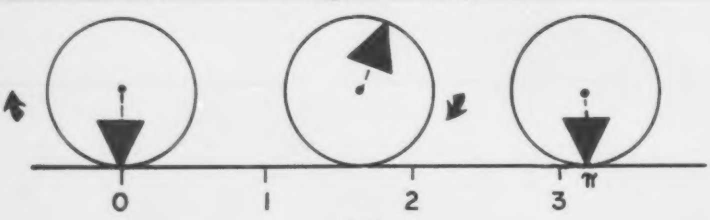
3. Since zero is not marked on this number line, you cannot perform any of the operations.

20-2. Locating Numbers on the Number Line.

In the preceding section the student learned that every point on the number line corresponds to a real number. In this section the student locates on the number line (by use of the Pythagorean Theorem studied in the last chapter) a point corresponding to a number $\sqrt{2}$ whose square is 2. In the next section the student will learn that this number $\sqrt{2}$ is not rational.

Although it is not related to the main purpose of the section, location on the number line of rational numbers (expressed as fractions) is also reviewed. The teacher is reminded of the "similar triangle" construction for locating such points given in Chapter 8 of the Teacher's Commentary.

The teacher may wish to present the plausible method of locating the number π by rolling (without slipping) a circle of diameter 1 along the number line.

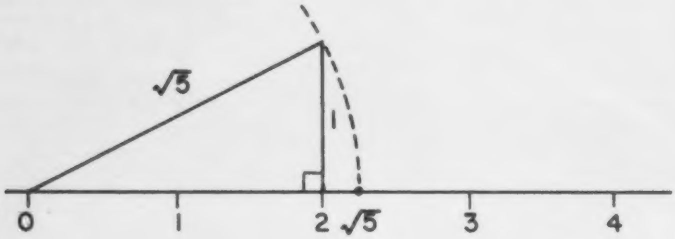


When the initial point of contact of the circle again touches the number line, then the circle has laid off its entire circumference, π, on the line.

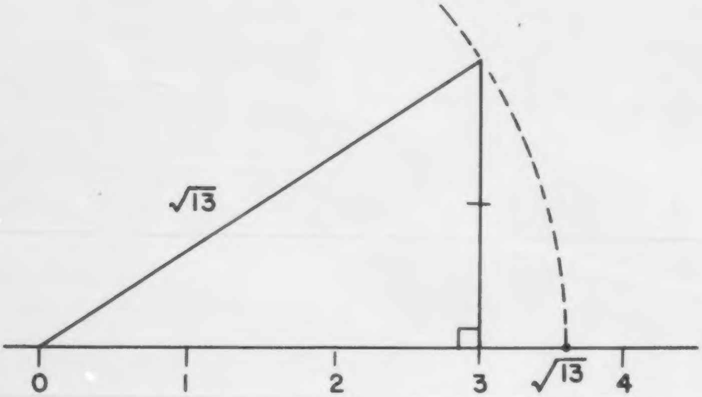
In Problem 6, Exercises 20-2, be sure to tell the student to place the figure properly on the sheet of paper so as not to run off the page. (See solution below.)

Answers to Exercises 20-2

1. The relation $1^2 + 2^2 = 5$ suggests a right triangle with legs 1 and 2 and hypotenuse $\sqrt{5}$.



2. Draw a right triangle with legs 2 and 3.



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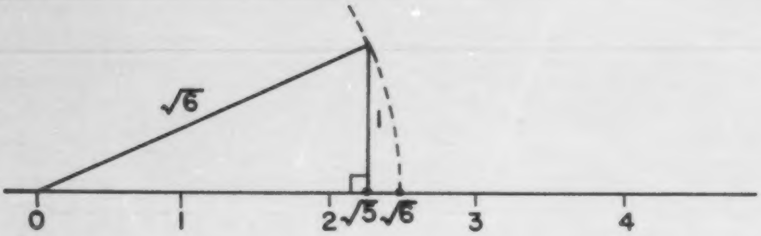
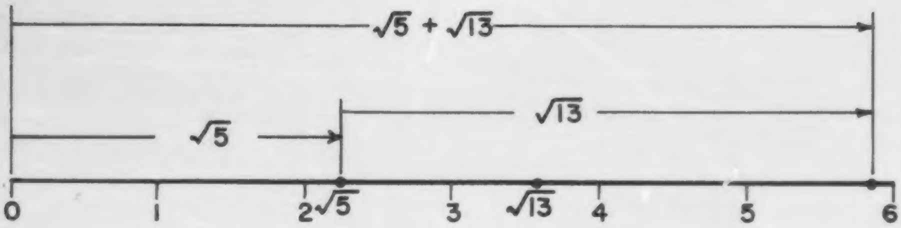
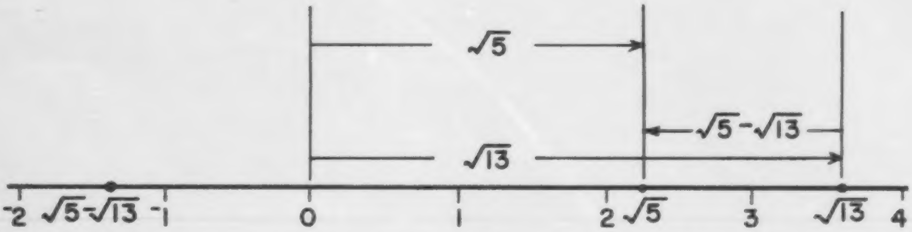
3. Draw a right triangle with legs 1 and $\sqrt{5}$.



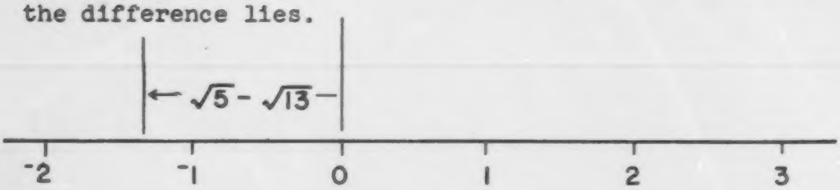
4.


√5 + √13
√13
√5
0
1
2
√5
3
√13
4
5
6

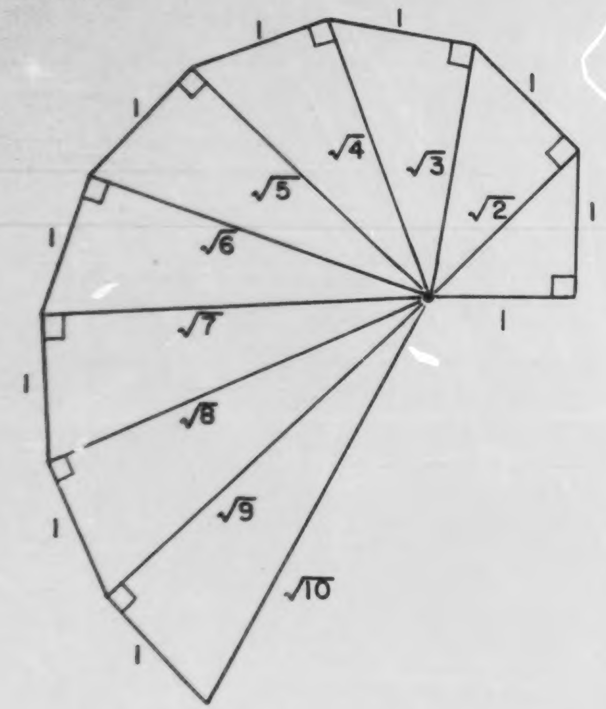
5.


√5
√5 - √13
√13
-2
√5 - √13
-1
0
1
2
√5
3
√13
4

In the diagram below the arrow $\sqrt{5} - \sqrt{13}$ is placed at the origin so you can see where the point representing the difference lies.



6.



20-3. Irrational Numbers.

The purpose of this section is to demonstrate the irrationality of $\sqrt{2}$. The traditional proof goes as follows:

> Suppose $\sqrt{2}$ is rational. Then $\sqrt{2}$ can be expressed as a quotient of integers, $\frac{a}{b}$. If a and b have factors in common, then we can reduce the fraction, repeated if necessary, until an equivalent fraction is obtained in which numerator and denominator have no factors in common.
>
> Accordingly, let p and q be whole numbers with no factors in common with
>
> $$\frac{p}{q} = \sqrt{2}$$
>
> then
>
> $$\frac{p^2}{q^2} = 2$$
>
> so that
>
> (1) $$p^2 = 2q^2 .$$

This last expression tells us that p^2 is an even number so that p must also be even (the square of an odd number is odd). Since p is even, there is a whole number r for which

$$(2) \qquad p = 2r.$$

Substituting (2) in (1) we see that

$$(2r)^2 = 2q^2$$

or

$$4r^2 = 2q^2$$

from which

$$2r^2 = q^2 .$$

This last expression tells us that q^2 is even, so that q is even whence for some whole number s we have

$$(3) \qquad q = 2s .$$

Now (2) and (3) show us that p and q have the common factor 2 in contradiction to the assumption that p and q have no factors in common. This means that there do not exist whole numbers p and q with no factors in common for which

$$\frac{p}{q} = \sqrt{2}.$$

But we have seen that if $\sqrt{2}$ is rational, then such numbers p and q must exist. Consequently, $\sqrt{2}$ cannot be rational.

Most students have a great deal of difficulty with this proof. In the first place, the idea of proof by contradiction causes trouble. But even those students who understand the idea of proof by contradiction have difficulty with this particular use of the idea. The reason is that only part of the statement "p and q are whole numbers having no factors in common satisfying $\frac{p}{q} = \sqrt{2}$ " has been contradicted, and a seemingly less important part at that, namely that p and q have no factors in common.

The student's reaction seems to be, "So what? If you cannot find whole numbers p and q without factors in common satisfying $\frac{p}{q} = \sqrt{2}$, then find them with factors in common." The teacher can then remind the student that an equivalent fraction could then be obtained in which numerator and denominator have no factors in common thus obtaining the case shown to be impossible. The student may not have an answer for this but he remains skeptical about the proof.

In the text an attempt has been made to modify the proof so as not to focus attention on the condition that p and q (a and b were used in the text) have no factors in common.

The proof follows these lines:
First, it is shown that if $\sqrt{2}$ is rational, then there will be whole numbers a and b meeting these requirements:

(i) $a^2 = 2b^2$,
(ii) a and b have no factors in common.

[The student has no objection at the onset to a and b having no factors in common. It is only after he sees that the whole proof rests on this point that he makes his objection.] Next an attempt is made to determine whether these requirements can possibly be met for each of the following oddness and evenness combinations. I (a even, b even), II (a even, b odd), III (a odd, b even), IV (a odd, b odd). All these cases are ruled out, and since these cases exhaust all possibilities, we see that there are no whole numbers a and b meeting requirements (i) and (ii) or, in short, that $\sqrt{2}$ is not rational.

We hope that the student will offer less resistance to this presentation than to the conventional one.

In the proof of the irrationality of $\sqrt{2}$ given in the student's text three facts are tacitly assumed to be part of the students' background in arithmetic. They are:

I. The product of two odd numbers is odd.
II. The product of two even numbers is a multiple of 4.
III. The product of two times an odd number is not a multiple of 4.

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It was felt that including proofs of these facts in the students' text would detract from the main ideas. For the teacher's benefit these proofs are given here.

I. If a and b are odd, then for some whole numbers m and n,

$$a = 2m + 1 \quad \text{and} \quad b = 2n + 1$$

so that

$$\begin{aligned} a \cdot b &= (2m + 1)\cdot(2n + 1) \\ &= 2m \cdot 2n + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

so that $a \cdot b$ is odd (a multiple of 2 plus 1).

II. If c and d are even, then for some whole numbers r and s

$$a = 2r \quad \text{and} \quad b = 2s,$$

so that

$$\begin{aligned} a \cdot b &= (2r)\cdot(2s) \\ &= (2 \cdot 2)\cdot(rs) \\ &= 4rs, \end{aligned}$$

so that $a \cdot b$ is a multiple of 4.

III. If $2 \cdot x$ is a multiple of 4, then for some whole number p

$$2 \cdot x = 4 \cdot p$$

from which

$$x = \frac{4 \cdot p}{2}$$

or

$$x = 2p \; .$$

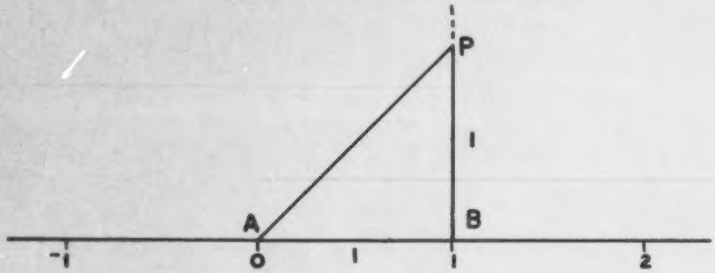
Therefore, x is even. Therefore, 2 times an odd number cannot be a multiple of 4.

The proof of II above provides the answer to a parenthetical question in the text.

In the text the remark is made that there are more irrational numbers than rational numbers. An explanation of this statement is given in this commentary in Section 20-4.

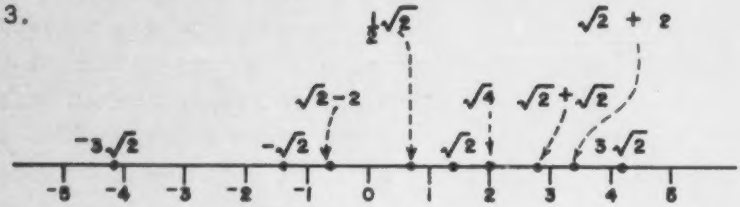
Answers to Exercises 20-3

1.



AP = $\sqrt{2}$

2. and 3.



4. All the points except $\sqrt{4}$ correspond to irrational numbers.

---

*20-4. Irrational Numbers and Infinite Decimals.

Without going into too much detail an effort is made to convince the student that each point on the number line has a unique infinite decimal representation. The student will probably readily accept the converse (that each infinite decimal represents a point on the number line) as being true also. However, this converse is much more difficult to establish rigorously, too difficult in fact to be included in this commentary.

One detail has been omitted as likely to cause confusion. It is that infinite decimals in which all digits are nines from some point on must be excluded if the correspondence between real numbers and infinite decimals is to be one-to-one. The reason for this is that, for example,

$$.3799999\ldots = .38000000\ldots .$$

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This may be seen as follows:

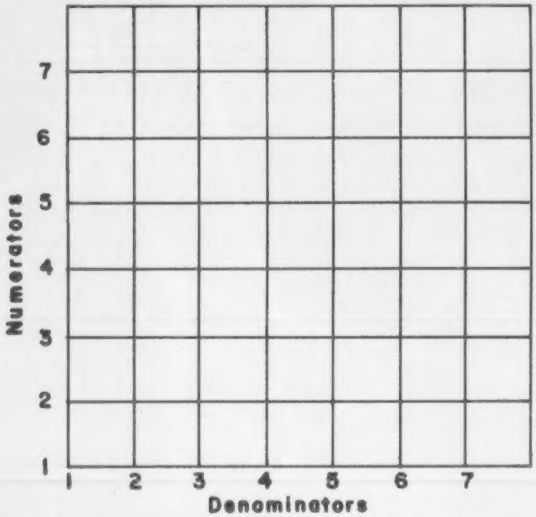
$$\text{If} \quad x = .37\overline{9} \quad \text{then} \quad 10x = 3.79\overline{9} \quad \text{so that}$$

$$\begin{array}{r} 10x = 3.79\overline{9} \\ \underline{\quad x = \ .37\overline{9}} \\ 9x = 3.42 \end{array} \quad \text{(By subtraction)}$$

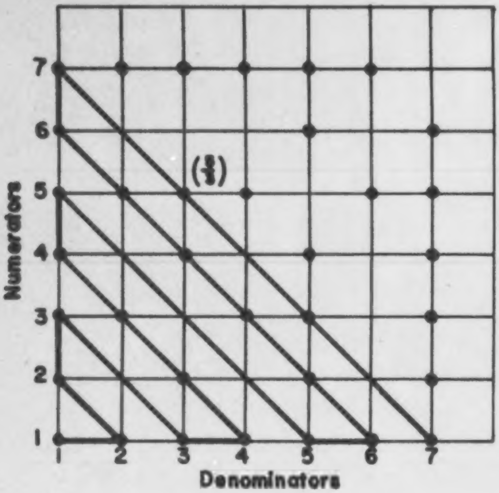
$$\text{so that} \quad x = \frac{3.42}{9} = .38 \ .$$

For better comprehension by the teacher of the real number system we include two demonstrations that are too difficult for almost all students. The exposition given here is necessarily quite condensed. Teachers wishing to read a more complete exposition of this material may find it in several of the references listed at the beginning of this chapter; for example, in Kasner and Newman, MATHEMATICS AND THE IMAGINATION. These are that (1) the rational numbers may be put into one-to-one correspondence with the integers, (2) the real numbers cannot be put into one-to-one correspondence with the integers.

To show (1), first make up a grid as follows:



Each intersection of lines on this grid denotes a fraction. Now mark with a heavy dot those intersections denoting fractions whose numerators and denominators have no factors in common.



Each positive rational number is then represented by one and only one black dot. Now the "zig-zag line" indicated in the figure above may be drawn. Following this "zig-zag line" we count the black dots as we come to them. This process of counting establishes a one-to-one correspondence between the counting numbers and the black dots (or positive rational numbers). The first few pairs in this correspondence are shown below:

1	2	3	4	5	6	7	8	9	10	11	12	13	...
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{2}{5}$	...

To show (2) that there are too many real numbers to be put into one-to-one correspondence with the counting numbers we must proceed by contradiction. And, in fact, we shall show even more; namely, that there are too many real numbers between 0 and 1 to be put into one-to-one correspondence with the counting numbers.

Suppose that the real numbers can be put into one-to-one correspondence with the counting numbers. Then we can make up an infinite list as shown below.

$$\begin{array}{ccl} 1 & \longleftrightarrow & .a_{11}a_{12}a_{13}a_{14}\cdots \\ 2 & \longleftrightarrow & .a_{21}a_{22}a_{23}a_{24}\cdots \\ 3 & \longleftrightarrow & .a_{31}a_{32}a_{33}a_{34}\cdots \\ 4 & \longleftrightarrow & .a_{41}a_{42}a_{43}a_{44}\cdots \\ \vdots & & \vdots\ \ \vdots \end{array}$$

The symbols a_{11}, a_{12}, a_{34}, a_{78}, etc., represent the digits in the infinite decimals. Here, for example, $.a_{31}a_{32}a_{33}a_{34}\cdots$ represents the infinite decimal representation of that real number which has been paired off with the counting number 3 in this one-to-one correspondence. (Note that the first subscript on each of the a's is 3.) The second subscripts on the a's indicate the position of the digit in the infinite decimal.

Now we can show how to construct an infinite decimal representing a number between 0 and 1 which cannot be in the list. Do it as follows. Choose digits b_1, b_2, b_3, etc., so that $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, $b_3 \neq a_{33}$, etc; also, in order to avoid the difficulty with nines mentioned earlier, be sure to choose all the digits b_1, b_2, b_3, etc., different from 9. Now the decimal

$$.b_1b_2b_3b_4\cdots$$

cannot be in the list because it differs from each decimal in the list in at least one "place". That is, it differs from $.a_{11}a_{12}a_{13}a_{14}\cdots$ in the first place, from $.a_{21}a_{22}a_{23}a_{24}\cdots$ in the second place, etc.

Therefore, we have a contradiction to the assumption that all the real numbers between 0 and 1 were included in this list. Therefore, no such list can be constructed. The set of real numbers between 0 and 1 cannot be put into one-to-one correspondence with the set of counting numbers.

This explains the remark in the student's text in Section 20-3 that there are more irrational numbers than rational numbers.

Answers to Exercises 20-4a

1. a. $9.9\overline{9}$

b. $312.12\overline{12}$

c. $35.035\overline{035}$

d. $166.6\overline{6}$

e. $0.04\overline{4}$

f. $613.45\overline{345}$

g. $803.15\overline{15}$

h. $31289.9\overline{9}$

i. $3128.9\overline{9}$

j. $60123.0123\overline{0123}$

2. a. $2816.10\overline{0}$ or 2816.1

b. $9.0\overline{0}$ or 9

c. $162.0\overline{0}$ or 162

d. $298.0\overline{0}$ or 298

e. $1.110\overline{0}$ or 1.11

f. $351.0\overline{0}$ or 351

g. $27048.0\overline{0}$ or 27048

h. $374.830\overline{0}$ or 374.83

3. a.

$$\begin{array}{rcl} 10(.5\overline{5}) &=& 5.5\overline{5}\\ .5\overline{5} &=& 0.5\overline{5}\\ \hline 9(.5\overline{5}) &=& 5.0\overline{0} \end{array}$$

$$.5\overline{5} = \frac{5}{9}$$

b.

$$\begin{array}{rcl} 100(.73\overline{73}) &=& 73.73\overline{73}\\ .73\overline{73} &=& 0.73\overline{73}\\ \hline 99(.73\overline{73}) &=& 73.0\overline{0} \end{array}$$

$$.73\overline{73} = \frac{73}{99}$$

c.

$$\begin{array}{rcl} 1000(.901\overline{901}) &=& 901.901\overline{901}\\ .901\overline{901} &=& 0.901\overline{901}\\ \hline 999(.901\overline{901}) &=& 901.0\overline{0} \end{array}$$

$$.901\overline{901} = \frac{901}{999}$$

*d.

$$\begin{array}{rcl} 10(3.023\overline{3}) &=& 30.233\overline{3}\\ 3.023\overline{3} &=& 3.023\overline{3}\\ \hline 9(3.023\overline{3}) &=& 27.210\overline{0} \end{array}$$

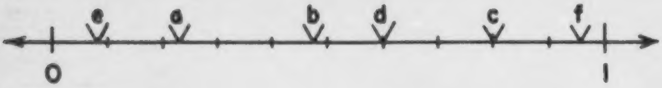
$$3.023\overline{3} = \frac{27.21}{9} = \frac{2721}{900} = \frac{907}{300}$$

Answers to Exercises 20-4b

1. a. 1.372 1.379 1.385 1.493 5.468
   b. $^{-}9.426$ $^{-}5.630$ $^{-}2.765$ 2.763 2.761
   c. $^{-}0.15475$ 0.15463 0.15467 0.15475 0.15598

2. a. none c. all but $^{-}0.15475$
   b. all but $^{-}0.15475$ d. all but $^{-}0.15475$
   e. 0.15475 0.15467 0.15463

3.



4. a. rational f. not enough information
   b. irrational g. irrational
   c. rational h. irrational
   d. rational i. rational
   e. irrational j. irrational

5. a. 2.996361
   b. 2.999824
   c. 3.003289
   d. .0003639 .000176 .003289
   e. 1.732

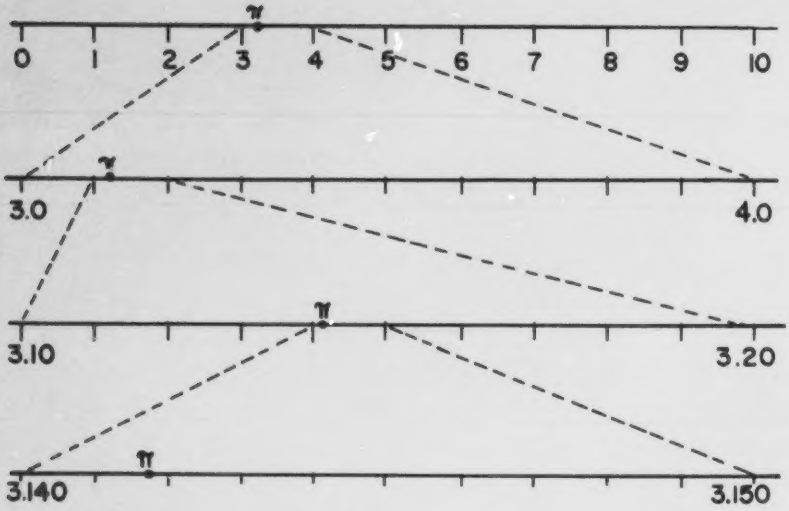
6. $(1.73)^2 = 2.9929$, $(1.74)^2 = 3.0276$, 1.73 is the better

7. $(3.87)^2 = 14.9769$, $(3.88)^2 = 15.0544$, 3.87 is the better

8. $(25.2)^2 = 635.04$, $(25.3)^2 = 640.09$, 25.2 is the better

9. $x = 6.473...$

10.


π
0
1
2
3
4
5
6
7
8
9
10
π
3.0
4.0
π
3.10
3.20
π
3.140
3.150

20-5. Properties of Number Systems.

Answers to Exercises 20-5

1. d and e

2. a. 2
   b. 2
   c. 2
   d. none

3. a. 5 (The counting numbers, 1, 2, 3, 4, 5)
   b. 6 (The whole numbers, 0, 1, 2, 3, 4, 5)
   c. 8 (Integers, $^{-}2$, $^{-}1$, 0, 1, 2, 3, 4, 5)
   d. infinitely many
   e. infinitely many

4. a. none
   b. 0
   c. 0
   d. none

5. a. yes
   b. no
   c. no
   d. no

20-7. Chapter Review.

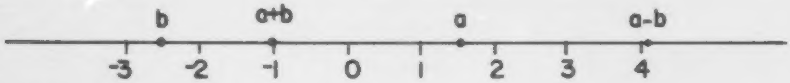
Answers to Exercises 20-7

1. a. closure (+, ×)
commutative (+, ×)
associative (+, ×)
identity +
distributive
order

b. closure ×
commutative ×
associative ×
identity ×
order

2. a. irrational - circumference is π units.
b. rational - area is 1 square unit.
c. rational - hypotenuse is 13 units.
d. rational - $(\sqrt{3})^2 = 3$.
e. irrational - volume is 2π cubic units.
f. rational - (each side is $\sqrt{2}$ units).

3.



4. $\frac{355}{113} = 3.1415929\ldots$

$\frac{22}{7} = 3.1428571\ldots$

$\pi = 3.1415926\ldots$

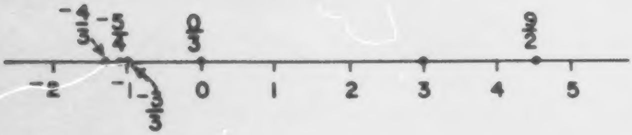
$\frac{355}{113}$ is a much better approximation to π than is $\frac{22}{7}$.

*5. $\frac{7}{11}$

20-8. Cumulative Review.

Answers to Exercises 20-8

1.



2. a. $\frac{1}{2}$ d. $\frac{8}{9}$
   b. $\frac{17}{12}$ or $1\frac{5}{12}$ e. $\frac{4}{9}$
   c. $^-(\frac{1}{12})$

3. a. $\frac{3}{5}$
   b. $\frac{63}{5}$
   c. $\frac{7}{200}$

4. a. 74844.55
   b. 105

5. $\frac{216}{126} < \frac{284}{161}$ since $216 \times 161 < 126 \times 284$
   or $34,776 < 35,784$

6. a. 5 c. $^-5$ e. 3
   b. $\frac{1}{5}$ d. 0 f. $^-(\frac{1}{3})$

7. a. $28 - x$
   b. $3x$
   c. $7(5 + x)$

8. a. 0, 90 b. 90, 180 c. 90

9. 70 cu. ft.

10. 15

11. Area $= \pi \cdot 2^2 - 2 \cdot \pi \cdot 1^2 = 2\pi \approx 6.28$
    The teacher should encourage the exact answer 2π sq. in. rather than the approximate answer 6.28 sq. in.

Sample Test Questions for Chapter 20

Note: Teachers should construct their own tests, using carefully selected items from those given here and from items of their own. Careful attention should be given to difficulty of items and time required to complete the test.

True or false:

(F) 1. Every real number can be written as a rational number.

(T) 2. The smallest positive integer is one.

(T) 3. $\sqrt{2}$ is a number which when squared is equal to 2.

(T) 4. Three and one-seventh is a rational number.

(T) 5. Every repeating decimal is a rational number.

(F) 6. The square root of 7 is approximately equal to 1.645.

(T) 7. Every real number can be represented by a point on the number line.

(F) 8. The number zero is not a rational number.

(F) 9. There are 12 integers between 15 and 27.

(F) 10. If a number is a real number then it is also a rational number.

(T) 11. A rational number may be expressed as an integer divided by a counting number.

(T) 12. $2\sqrt{2}$ is both a real number and an irrational number.

(F) 13. Irrational numbers cannot be located on the number line.

(F) 14. Zero is a number that is both rational and irrational.

Multiple Choice:

1. The number $\sqrt{2}$ may be classified as:

A. real and irrational.
B. real and rational.
(A) C. rational but not real.
D. irrational but not real.
E. both rational and irrational.

2. $\sqrt{7}$ lies between:

A. 2.1 and 2.2
B. 2.3 and 2.4
(D) C. 2.5 and 2.6
D. 2.6 and 2.7
E. 2.7 and 2.8

3. Which of the following is a rational number?

A. $\sqrt{2}$
B. $\sqrt{3}$
(C) C. $\sqrt{4}$
D. $\sqrt{5}$
E. $\sqrt{6}$

4. Which of the following is NOT a real number?

A. 0
B. 12
(E) C. $7\frac{1}{2}$
D. $.09\overline{09}$
E. All of the above are real numbers.

5. In which of the following sets of numbers is there always another number of the set between any two given numbers of the set?

A. Prime numbers
B. Rational numbers
(B) C. Counting Numbers
D. Whole Numbers
E. Integers

6. The set of integers is included in the set of:

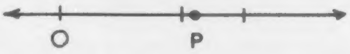
A. Counting numbers
B. Whole numbers
(D) C. Non-negative numbers
D. Real numbers
E. Positive rational numbers

7. There is a one-to-one correspondence between all the points on the number line and the elements of which of the following sets of numbers?

A. Counting numbers
B. Integers
(E) C. Rational numbers
D. Irrational numbers
E. Real numbers

8. On the number line at the right, P represents any point. We can be certain that P represents:

A. A real number
B. A rational number
(A) C. An irrational number
D. An integer
E. A whole number



9. Which of the following is an irrational number?

A. The ratio of the measure of the perimeter of a square to the measure of a side.
B. The ratio of the measure of the circumference of a circle to the measure of the diameter.
(B) C. $\frac{a}{b}$, where a and b are integers, $b \neq 0$.
D. The number of pennies in a dollar.
E. The value of P in $P = 3s$ when $s = 2$.

Chapter 21

SCIENTIFIC NOTATION, DECIMALS, AND THE METRIC SYSTEM

In this chapter we assume that the student has had some acquaintance with the names of numbers, the decimal notation, and finding products involving decimals and percents.

We intend that the class discussion exercises will be done during class time. The procedure for getting each answer should be discussed before the class continues to the next problem.

This chapter should take about 13 days.

21-1. Large Numbers and Scientific Notation.

This section seeks to cultivate the ability to read large numbers, the appreciation of them, and the ability to write them in scientific notation. (Some people prefer to use the term "standard form" instead of "scientific notation.") A certain amount of estimation is also included.

Of course there is no largest number, and two knowledgeable boys would play the game described to a draw by the simple process of adding one, multiplying by two or in some other way increasing the number which their opponent had just given.

Some of the more thoughtful students may wonder why we do not write 93 million, for instance, as 93×10^6 where the exponent is used to indicate the number of zeros in the numeral. There is no point in trying to hide the fact that in many cases this is really a little simpler. You can call this exponential notation.

Answers to Exercises 21-1a

1. a. 1,000,000,000

   b. 1,000,000,000,000

   c. 1,000,000,000,000,000

   d. 1,000

   e. 10,000

   f. 100

2. a. 10^9
   b. 10^{12}
   c. 10^{15}
   d. 10^3
   e. 10^4
   f. 10^2

3. a. 7×10^3
   b. 5×10^4
   c. 3×10^6
   d. 14×10^6
   e. 375×10^6
   f. 48×10^{10}

It is also correct to have other answers than the above. For example, (b) might be expressed as 50×10^3, 500×10^2, 5000×10^1, or even $50{,}000 \times 10^0$.

4. a. 100 or 10^2
   b. 10,000 or 10^4
   c. 830
   d. 83
   e. 100,000 or 10^5
   f. 57,000
   g. 57
   h. 10,000 or 10^4
   *i. 4.20 or 4.2
   *j. 3.21

The term, scientific notation, is in common usage and is a convenient one. As students learn about exponential notation, it is useful to have a unique form rather than many for each answer. The teacher should realize that it is not the only notation used in science, and that the term is not even widely known to scientists. Exponential notation, however, is frequently used.

Answers to Exercises 21-1b

1. a. 7.6×10
   b. 8.59×10^2
   c. 7.623×10^3
   d. 8.463×10^3
   e. 7.64×10
   f. 4.835×10^2
   g. 8.412×10^2
   h. 9.7836×10^3
   i. 3.412789435×10^6

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2. a. No, because 15 is not between 1 and 10.
   b. Yes, it satisfies the definition.
   c. No, because 12.0 is not between 1 and 10.

3. a. 5.687×10^3
   b. 1.4×10
   c. 3.5×10^6
   d. 2.7×10
   e. 6.13×10^2
   f. 2.05×10^2

4. a. 3,700,000
   b. 470,000
   c. 5,721,000
   d. 10,000
   e. 1600
   f. 8,300,000,000

5. a. 6000
   b. 7000
   c. 675,000
   d. 7000 or 6000
   e. 7000
   f. 6000

Answers to Exercises 21-1c

1. a. 10^2
   b. 10^3
   c. 10^4
   d. 6.87×10^2
   e. 6×10^3
   f. 7.832×10^3
   g. 10^6
   h. 7.81×10^9
   i. 6×10^3
   j. 9×10^9

2. a. 1000
   b. 100,000
   c. 583
   d. 30,000
   e. 630
   f. 436,000,000
   g. 1,000,000,000
   h. 1,730,000

3. a. Seven hundred eighty-three
   b. Seven million, five hundred thousand
   c. Sixty-three thousand seven
   d. Three hundred sixty-two and thirty-six hundredths
   e. Two hundred eighty-four and sixty-three hundredths
   f. Four and two hundred fifty-six thousandths

4. a. 600
   b. 100
   c. 1200
   d. 70900
   e. 600
   f. 362,400

5. a. 6×10^2
   b. 10^2
   c. 1.2×10^3
   d. 7.09×10^4
   e. 6×10^2
   f. 3.624×10^5

21-2. Multiplying by Powers of Ten.

The objectives of this section are a continuation of those in the first with the added skill of multiplying, using scientific notation.

Some students may prefer to multiply 93,000,000 by 11,000, for example, by writing the former as 93×10^6 and the latter as 11×10^3, then forming the product 1023×10^9, and then putting it into scientific notation: 1.023×10^{12}. The teacher may prefer this, too.

To show that $1.023 \times 10^3 = 1023$ one can refer to the rules for multiplying decimals or the teacher may prefer to go back to first principles. One way to do the latter is:

$$1.023 = 1 + \frac{23}{1000} \text{ and, since } 10^3 = 1000,$$

$$\begin{aligned} 1.023 \times 10^3 &= \left(1 + \frac{23}{1000}\right) \times 1000 \\ &= 1000 + \frac{23}{1000} \times 1000 \\ &= 1000 + 23 \\ &= 1023. \end{aligned}$$

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The student should realize that multiplying a number by 10 is equivalent to moving each digit one place to the left.

Answers to Exercises 21-2a

1. a. 10^{10}
   b. 10^{5}
   c. 10^{10}
   d. 10^{11}
   e. 10^{7}
   f. 10^{20}
   g. 6×10^{10}
   h. 10^{19}
   i. 10^{6}
   j. 10^{6}

2. a. 8×10^{5}
   b. 5×10^{7}
   c. 2.3×10^{5}
   d. 4×10^{6}
   e. 5.4×10^{5}
   f. 10^{6}
   g. 4×10^{8}
   h. 3.7×10^{8}
   i. 5.12×10^{5}
   j. 4×10^{7}
   k. 6.72×10^{9}
   l. 4.5×10^{9}

3. a. 1.2×10 miles
   b. 7.2×10^{2} miles

4. a. 3.6×10^{5} miles
   b. 4.32×10^{7} miles
   c. 3.1536×10^{8} miles
   d. Yes

Answers to Exercises 21-2b

1. a. 6×10^{10}
   b. 1.2×10^{18}
   c. 3.5×10^{13}
   d. 3×10^{7}
   e. 7.63×10^{7}
   f. 2.16×10^{5}
   g. 9.3×10^{8}

2. a. 6.3×10^{11}
   b. 10^{9}
   c. 4.65×10^{9}
   d. 1.1×10^{9}

3. a. 4×10^7
   b. 2×10^{10}
   c. 1.2×10^6

4. 7.2×10^9 miles

5. About 6.132×10^8 miles or 613,200,000 miles.

6. BRAINBUSTER: Yes, you can make 63,072,000 marks.
   $60 \times 60 \times 24 \times 365$ seconds is one year.

21-3. Dividing by Powers of Ten.

To multiply numbers expressed as powers of 10 we add the exponents and express the product as a power of 10 with this sum as the exponent.

This class discussion should lead the pupils to discover the division operation of numbers expressed in exponential form. If the examples in the text are not sufficient for this purpose, use additional examples until the students discover that in dividing numbers written as powers of 10, the exponent in the denominator is subtracted from the exponent in the numerator.

$$\frac{10^a}{10^b} = 10^{a - b}$$

Answers to Exercises 21-3a

1. 10^3
2. 10^4
3. 10^2
4. Perhaps several students will have discovered the method of subtracting exponents at this point, namely that $10^a \div 10^b = 10^{a-b}$

5. 10^6

6. 10^3, 10^2, 10^1 ; $\frac{10}{10} = 1$ or 10^0 (This gives motivation for the next portion of the text.)

We are extending the meaning of an exponent from a counting number to an integer. The idea of an exponent as the number of times the base appears as a factor in a product becomes meaningless when the exponent is 0 or a negative integer. Instead we choose our definitions to preserve certain properties that exponents have. This is similar to our treatment of the rational numbers as an extension of the whole numbers. The most important property is that $10^a \cdot 10^b = 10^{a+b}$. We want this property to hold even when a and b are not counting numbers. This is the reason behind our definition of 10^0 as 1 and 10^{-n} as $\frac{1}{10^n}$. We have just tried to make the definitions reasonable to the student in a very intuitive way. Later in algebra the meaning of exponent is extended to rational numbers and still later to real numbers.

Answers to Exercises 21-3b

1. a. 10^{-4}
   b. 10^{-6}
   c. 10^{-7}
   d. 10^{-2}
   e. 10^{-9}
   f. 10^{-5}
   g. 10^{-3}
   h. 10^{-19}
   i. 10^{-1}
   j. 10^{-4}
   k. 10^{-3}
   l. 10^{-5}

2. a. $\frac{1}{10^3}$ or $\frac{1}{1000}$
   b. $\frac{1}{10^5}$ or $\frac{1}{100,000}$
   c. $\frac{1}{10^7}$ or $\frac{1}{10,000,000}$
   d. $\frac{1}{10^6}$ or $\frac{1}{1,000,000}$
   e. $\frac{1}{10^{27}}$
   f. $\frac{1}{10^{11}}$
   g. $\frac{1}{10^{256}}$
   h. $\frac{1}{10^{529}}$

3. a. .01
   b. .0001
   c. .1
   d. .000001

4. a. 10^{1}
   b. 10^{-1}
   c. 10^{1}
   d. 10^{-2}
   e. 10^{0}
   f. 10^{3}
   g. 10^{-3}
   h. 10^{-1}
   i. 10^{-3}

Answers to Exercises 21-3c

1. a. 9.3×10^{-2}
   b. 10^{-4}
   c. 10^{-6}
   d. 10^{0}
   e. 6.21×10^{-3}
   f. 10^{-2}
   g. 7.006×10^{-1}
   h. 9.07×10^{-7}
   i. 6×10^{0}
   j. 4.5×10^{-3}

2. a. 0.000093
   b. 0.107
   c. 0.000001
   d. 0.0005
   e. 0.007065
   f. 0.1
   g. 0.00000143
   h. 0.00038576

3. a. 6.3×10^{5}
   b. 1.57×10^{-4}
   c. 2.4×10^{-6}
   d. 5.265×10^{-5}
   e. 3.6235×10^{2}
   f. 4.32×10^{-3}
   g. 3.05×10^{-9}
   h. 6.95×10^{0}

4. a. $^{-}3$
   b. 6.3
   c. $^{-}7$
   d. 5
   e. $^{-}1$
   f. $^{-}3$
   g. 21300
   h. 2130

5. a. >
   b. <
   c. <
   d. >
   e. <
   f. >
   g. <
   h. <
   i. >
   j. <

6. a. ii
   b. iii
   c. i
   d. i
   e. ii
   f. i
   g. iii
   h. i
   i. i
   *j. ii
   *k. ii
   *l. i

21-4. Multiplication of Large and Small Numbers.

The development in this section will lead the students, in easy steps, to the conclusion that:

$$10^a \times 10^b = 10^{a+b}$$

where a and b are integers.

Answers to Exercises 21-4a

1. a. Definition of $10^{^-n}$.
   b. Product of rational numbers expressed as fractions.
   c. Multiplication of powers of 10, discovered in Section 21-1.
   d. Definition of $10^{^-n}$.
   f. $^-8$
   g. We hope for a statement that when multiplying powers of 10 you can express the product as a power of 10 where the exponent is the sum of the exponents of each factor. Students noticed that this was true when the exponents were counting numbers and now it is still true when the exponents are integers. (This is the property of exponents we want to preserve and is the reason behind our definitions of 10^0 and $10^{^-n}$.)

2. 10^{-6}

3. a. Definition of 10^{-n}.
   b. $\frac{10^5}{10^3}$. Product of rational numbers expressed as fractions.
   c. 10^2. Division of powers of 10 discussed in Section 21-3.
   d. 10^2
   e. $5 + ({}^-3)$

4. a. Definition of 10^{-n}.
   b. $\frac{10^3}{10^5}$. Multiplication of rational numbers.
   c. $\frac{1}{10^2}$
   d. 10^{-2}
   e. 10^{-2}
   f. $3 + ({}^-5) = {}^-2$

5. a. 10^{-3}
   b. 10
   c. 10^7
   d. 10^{-6}
   e. 10^{-3}
   f. 10^{-1}

Answers to Exercises 21-4b

1. a. 10^{-7}
   b. 3×10^{-3}
   c. 10^{-13}
   d. 8×10^{-5}
   e. 7×10^{-7}
   f. 5.7×10^{-10}
   g. 10^{24}
   h. 10^{-3}

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2. a. 2.88×10^{-8} d. 3×10^{-10}

b. 5.4×10^{-9} e. 4.56×10^{-6}

c. 1.4×10^{-7} f. 5.6896×10^{-6}

3. a. 10^{2} c. 10^{-6}

b. 10^{2} d. 10^{-7}

4. 3.4594×10^{-1}

5. 4.125×10^{5} dollars

6. 9.939×10^{9} dollars

21-5. Division of Large and Small Numbers.

Answers to Exercises 21-5a

1. a. 10^{-9}
   b. 10^{5}

2. a. 10^{5}
   b. Because 7 - 2 equals 5
   c. The same number

3. a. 9
   b. 10^{9}
   c. Both equal 10^{9}

4. a. 10^{-9}
   b. Yes. Both equal 10^{-9}.

5. Yes. Each side equals 10^{5}.

Many students will have to review subtraction with negative numbers which they studied in Chapter 17.

$$\begin{aligned}
(3.6 \times 10^{-4}) \div (2 \times 10^{8}) &= \frac{3.6 \times 10^{-4}}{2 \times 10^{8}} && \text{Meaning of division} \\
&= \frac{3.6}{2} \times \frac{10^{-4}}{10^{8}} && \text{Multiplication of rational numbers} \\
&= 1.8 \times 10^{-4-8} && \text{By method developed in this section for} \\
&= 1.8 \times 10^{-12} && \text{dividing powers of 10.}
\end{aligned}$$

Answers to Exercises 21-5b

1. a. 10^{3}
   b. 10^{2}
   c. 10^{10}
   d. 10^{5}
   e. 10^{-2}
   f. 10^{-10}
   g. 10^{-6}
   h. 10^{-1}

2. a. 10^{7}
   b. 10^{4}
   c. 10^{18}
   d. 10^{29}
   e. 10^{24}
   f. 10^{30}
   g. 10^{18}
   h. 10^{7}

3. a. 10^{-7}
   b. 10^{-4}
   c. 10^{-18}
   d. 10^{-29}
   e. 10^{-24}
   f. 10^{-30}
   g. 10^{-18}
   h. 10^{-7}

4. a. 10^{3}
b. 10^{-10}
c. 10^{2}
d. 10^{6}
e. 10^{4}
f. 10^{-5}
g. 10^{10}
h. 10^{1}

5. a. 2×10^{-3}
b. 7×10^{-7}
c. 1.2×10^{9}
d. 2.4×10^{2}
e. 4×10^{-6}
f. 4×10^{-3}

6. a. $^{-}2$ and $^{-}1$
b. $^{-}2$ and $^{-}1$
c. 2, $^{-}2$, $^{-}2$
d. $^{-}3$ and $^{-}1$
e. $^{-}1$ and 1
f. $^{-}1$, 3, $^{-}1$, and 4

7. 9.2×10^{7}. Treat as problem in equations,

$$\frac{3}{100}y = 2.76 \times 10^{6}$$

8. No, since it will take 100 years to spend this sum of money.

*9. About 1180 days. (Rounded to nearest ten)

*10. \$20,000. Treat as problem in ratio.

*11. 40%. Treat as problem in equations,

$$14 \times 10^{6} = \frac{y}{100}(35 \times 10^{6})$$

21-6. Use of Exponents in Multiplying and Dividing Decimals.

Note that division is handled somewhat differently here. Powers of 10 are used in such a fashion that a whole number is divided by a whole number. Many people find this easier to follow than the procedure which uses a decimal as the dividend.

There are too many problems in the exercises to be given as one assignment. Some of the students may need the extra practice and will need two or more days.

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If your class is uncertain as to why multiplying or dividing by powers of 10 changes the place value of the digits, you may wish to return to expanded notation and review topics from Chapter 9, in Volume I.

Answers to Exercises 21-6a

1. $6.14 \times 0.42 = (614 \times 10^{-2}) \times (42 \times 10^{-2})$
$= (614 \times 42) \times (10^{-2} \times 10^{-2})$
$= 25788 \times 10^{-4}$
$= 2.5788$

2. $0.625 \times 0.038 = (625 \times 10^{-3}) \times (38 \times 10^{-3})$
$= (625 \times 38) \times (10^{-3} \times 10^{-3})$
$= 23750 \times 10^{-6}$
$= 0.023750$

3. $649.3 \times 14.68 = (6493 \times 1468) \times (10^{-1} \times 10^{-2})$
$= 9531724 \times 10^{-3}$
$= 9531.724$

4. $11.4 \times 0.0031 = (114 \times 31) \times (10^{-1} \times 10^{-4})$
$= 3534 \times 10^{-5}$
$= 0.03534$

Answers to Exercises 21-6b

1. a. 0.18063
   b. 0.0684
   c. 7500
   d. 399.529
   e. 7.2

2. a. $^{-}2$
   b. $^{-}3$
   c. 4
   d. $^{-}3$
   e. 63700
   f. 2, 1
   g. 0.0412

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3. a. 300200 c. 0.000007
b. 6.1 d. 160

Answers to Exercises 21-6c

1. a. $135 \times 6 \times 10^{-2} = 810 \times 10^{-2} = 8.1$

b. $(76 \times 10^{3}) \times (3 \times 10^{3}) = 228 \times 10^{6} = 228{,}000{,}000$

c. $(18 \times 10^{3}) \times (3 \times 10^{-4}) = 54 \times 10^{-1} = 5.4$

d. $(35 \times 10^{-4}) \times (16{,}301 \times 10^{-3}) = 570535 \times 10^{-7}$
$= 0.0570535$

e. $(6 \times 10^{6}) \times (275 \times 10^{-4}) = 1650 \times 10^{2} = 165{,}000$

f. $(7 \times 10^{-2}) \times (3 \times 10^{2}) \times (2 \times 10^{-2}) \times (6 \times 10^{3})$
$= 252 \times 10^{1} = 2520$

2. a. $(63 \times 10^{-1}) \div (3 \times 10^{-1}) = 21$

b. $(78 \times 10^{-2}) \div (13 \times 10^{0}) = 6 \times 10^{-2} = 0.06$

c. $\dfrac{8750 \times 10^{0}}{875 \times 10^{-2}} = 10 \times 10^{2} = 1000$

d. $\dfrac{1470 \times 10^{-4}}{75 \times 10^{-2}} = 19.6 \times 10^{-2} = 0.196$

e. $27 \times 10^{-2}\,\overline{)\,84402 \times 10^{-5}} = 3126 \times 10^{-3} = 3.126$

f. $18 \times 10^{2}\,\overline{)\,216 \times 10^{-1}} = 12 \times 10^{-3} = 0.012$

*3. $\dfrac{6 \times 840}{.04} = \dfrac{5040 \times 10^{0}}{4 \times 10^{-2}} = 1260 \times 10^{2} = 126000$

It is necessary to make 126,000 pieces.

4. Brainbuster:

$$\frac{5\frac{1}{3} \times (6.3 \times 10^{12})}{100,000 \times (3.2 \times 10^{7})} = \frac{\frac{16}{3} \times (63 \times 10^{13})}{10^{5} \times 32 \times 10^{8}} = \frac{16 \times 21}{32} = 10.5$$

It will take about 10.5 years to go one way, 21 years round trip.

21-7. The Metric System.

"The invention of the Hindu-Arabic decimal system of numeration is one of man's outstanding achievements. With it, for the first time in history, masses of people were able to learn the art of computation. Later Simon Stevin simplified the processes of computation still further by the introduction of the decimal fraction. Today, the decimal fraction should be called the common fraction, so widely is it used in commerce and technology.

"Still later came the metric system of measures, based upon the units, meter, liter, and gram, which are also decimal. ...If the selection of a system of measures were optional with educators, they would unhesitatingly choose a decimal system. They are well aware of the tremendous efforts required to learn, for example, the relationships among linear units in our system: 1 inch = $\frac{1}{12}$ foot, 1 foot = $\frac{1}{3}$ yard, 1 yard = $\frac{2}{11}$ rod, 1 rod = $\frac{1}{320}$ mile. In contrast, they appreciate the simplicity and ease with which the pupil can learn: 1 millimeter = 0.1 centimeter, 1 centimeter = 0.01 meter and 1 meter = 0.001 kilometer.

"From the point of view of teaching and learning, it would not be easy to design a more difficult system than the English system. In contrast, it would seem almost impossible to design a system more easily learned than the metric system."[1]

---

[1] Clark, John R., "A Note on the Yearbook," The Metric System of Weights and Measures, Twentieth Yearbook of National Council of Teachers of Mathematics, Bureau of Publication, Teachers College, Columbia University, New York, 1948.

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The above quotation from the Twentieth Yearbook of the National Council of Teachers of Mathematics served as a motivation for this section.

The brief historical sketch is intended to help the pupils see the decimal foundation and origin of the metric system.

Although the pupils have had a brief introduction to the metric system in the 7th grade, we urge you to have them "live metric" for a few days. Postpone any attempts at converting from metric to English measures. The pupils should have a metric ruler at hand in order to get well acquainted with the linear units.

Because of the length of this chapter we are able only to introduce the linear metric scale. We have not developed in any detail the two systems now in vogue, namely, the MKS (meter, kilogram, second) or the CGS (centimeter, gram, second).

We highly recommend the Twentieth Yearbook of the National Council of Teachers of Mathematics as a rich reference text for both your professional library and your pupils' mathematics library.

The original definition of the meter was used until October 15, 1960 when delegates from 32 nations agreed on a new standard definition. The meter is now defined in terms of the orange-red wave-lengths of Krypton gas, and one meter is defined as:

> 1,650,763.73 orange-red wave-lengths, in a vacuum, of an atom of the gas, Krypton 86.

This new definition has the advantage that it can be measured accurately by an interferometer anywhere in the world. The method of using a standard bar of platinum-iridium was not as precise.

Answers to Exercises 21-7a (Class Discussion)

1 and 2. There is no standard answer to these questions, but many interesting opinions should come from the class and stimulate interest in the metric system. (As we go to press Great Britain is in the process of changing to the metric system in order to enter the Common Market. You may find current news articles on this topic.)

Answers to Exercises 21-7b

1. The place value of each digit of the measurement expressed in the larger unit is $\frac{1}{10}$ as large. The new numeral can be written by moving the digits one place to the right.

2. The place value of each digit of the measurement expressed in the smaller is 10 times as large. The new numeral can be written by moving the digits one place to the left.

3. a. 10
   b. 100
   c. 1,000
   d. 10,000
   e. 100,000
   f. 1,000,000

4. a. 500
   b. 2000
   c. 0.5
   d. 25.4
   e. 1500
   f. 325
   g. 3.5
   h. 4.74
   i. 55
   j. 625

*5. a. 4×10^7
   b. 4×10^6
   c. 4×10^5

Answers to Exercises 21-7c

1. $11.11 \times 10^{-3} = 0.01111$
2. $5.34 \times 10^{2} = 534$
3. $24.5 \times 10^{-3} = 0.0245$
4. $0.52 \times 10^{3} = 520$
5. $643.2 \times 10^{-3} = 0.6432$
6. $202.2 \times 10^{2} = 20,220$
7. $0.015 \times 10^{3} = 15$

21-8. Conversion to English Units.

The ratio conversion method was chosen as a reinforcement of what the student learned in Chapter 10.

Answers to Exercises 21-8a

1. Solve the proportion by the Comparison Property (Chapter 10), $x = 22.86$ cm.
2. $\frac{1}{2.54} = \frac{15}{x}$; $x = 38.10$
3. $\frac{1}{2.54} = \frac{y}{27.94}$; $y = \frac{27.94}{2.54}$; $y = 11$
4. $\frac{1}{2.54} = \frac{y}{12.70}$; $y = \frac{12.70}{2.54}$; $y = 5$
5. Find the number of inches in 5 feet, then write
   $\frac{1}{2.54} = \frac{60}{x}$; $x = 152.40$
6. Find the number of inches in 9 yards, then write
   $\frac{1}{2.54} = \frac{324}{x}$; $x = 822.96$

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Answers to Exercises 21-8b

1. a. 20.32
   b. 60.96
   c. 48.26
   d. 68.58
   e. 274.32
   f. 91.44
   g. 457.20
   h. 731.52

2. a. 2
   b. 14
   c. 7
   d. 8.5
   e. 12
   f. 15
   g. 3
   h. 5.5

3. a. ≈ 110
   b. ≈ 220
   c. ≈ 440
   d. ≈ 880
   e. ≈ 1650

4. a. 6 mi.
   b. 30 mi.
   c. 60 mi.
   d. 300 mi.
   e. 600 mi.

5. Approximately 8839 meters

6. Approximately 155,000,000 km.

7. 21.59 cm. by 27.94 cm.

8. Answer depends on student's height.
   A boy 5'2" or 62" = 1574.8 mm.
   = 157.48 cm.
   = 1,574,800 microns

21-10. Chapter Review.

Answers to Exercises 21-10

1. a. 5×10^{2}
   b. 5×10^{3}
   c. 5×10^{4}
   d. 5.67×10^{5}
   e. 5.67×10^{8}
   f. 5.67×10^{11}

2. a. 10,000
   b. 483
   c. 300,000
   d. 930
   e. 536,000,000
   f. 1,530,000

3. a. 10^{5}
   b. 6×10^{9}
   c. 10^{17}
   d. 10^{0} or 1
   e. 10
   f. 10^{10}

4. a. 10^{-3}
   b. 10^{-2}
   c. 10^{-9}
   d. 10^{-4}
   e. 10^{-9}
   f. 10^{-19}

5. a. $\frac{1}{10}$
   b. $\frac{1}{10^{7}}$
   c. $\frac{1}{10^{28}}$
   d. $\frac{1}{10^{10}}$
   e. $\frac{1}{10^{297}}$
   f. $\frac{1}{10^{792}}$

6. a. 1.0×10^{-3}
   b. 1.2×10^{-4}
   c. 10^{-3}
   d. 2×10^{-9}
   e. 10^{5}
   f. 10^{14}

7. a. 10^{6}
   b. 10^{16}
   c. 10^{7}
   d. 10^{5}
   e. 10^{9}
   f. 10^{54}

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8. a. 400 cm.
   b. 1000 mm.
   c. .5 km.
   d. .0254 m.
   e. 54 mm.
   f. 525 cm.

21-11. Cumulative Review.

Answers to Exercises 21-11

1. 38 points

2. a. $\frac{17}{12}$ or $1\frac{5}{12}$
   b. $\frac{1}{2}$
   c. $\frac{8}{9}$
   d. $\frac{4}{9}$
   e. 0
   f. 1
   g. $\frac{1}{6}$
   h. $^-\frac{1}{6}$

3. $\frac{13}{42} > \frac{17}{56}$ since $13 \times 56 > 17 \times 42$
   or $728 > 714$

4. $^-4$

5. a. $^-12$
   b. $^-3.6$
   c. 3
   d. $^-5$
   e. $^-16$
   f. 3
   g. $^-\frac{1}{8}$
   h. $\frac{5}{2}$ or $2\frac{1}{2}$

6. a. 9
   b. 11
   c. $\frac{3}{7}$
   d. 0
   e. $^-14$
   f. $\frac{24}{11}$ or $2\frac{2}{11}$

7. 7 hr. 35 min.

8. $A = 34$ The area is 34 sq. ft.

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9. a. 4567.22 sq. ft. c. $\frac{9}{8}$ or $1\frac{1}{8}$ sq. ft.

b. 655.36 sq. cm. d. 1.376 sq. mm.

10. $75 = \pi d$

diameter $= \frac{75}{\pi}$ in. $\approx$ 24 in.

length of spoke $\approx$ 12 in.

11. More, since $365 \times 3\frac{1}{3} \times 10^6 > 10^9$

Sample Test Questions for Chapter 21

This set of questions is not intended as a chapter test. Teachers should construct a chapter test carefully by combining selected items from this set of questions and questions of their own writing. Care should be exercised to avoid making the test too long.

True-False:

(F) 1. In the symbol 10^3 the exponent is 10.
(F) 2. $10^2 \times 10 = 10^2$
(T) 3. $10^5 \times 10^{-5} = 1$
(T) 4. $1{,}000{,}000 = 10^6$
(T) 5. $1\frac{1}{2}\% = 1.5 \times 10^{-2}$
(T) 6. $\frac{1}{10^5} = \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10}$
(T) 7. 2.76 billion is 2.76×10^9
(F) 8. $10^3 = 10^{-2} + 10^5$
(F) 9. $93{,}000{,}000 = 9.3 \times 10^6$
(F) 10. $10^3 \times 10^3 = 10^9$
(T) 11. 10^4 means $10 \times 10 \times 10 \times 10$
(F) 12. 11×10^7 is written in scientific notation.

Completion:

1. The closest star to us is 24,500,000,000,000 miles away. Write this distance using scientific notation. (2.45×10^{13})

2. Perform the indicated operations and give the answer in scientific notation.
   a. $(7 \times 10^{-7}) \times 10^{17}$ (7×10^{10})
   b. $6 \times 10^{-5} \times 9 \times 10^{-3}$ (5.4×10^{-7})
   c. 19.6×0.028 (5.488×10^{-1})
   d. $0.787 \times 200{,}000$ (1.574×10^{5})

3. Fill in the blanks with the correct power of ten.
   a. $7.81 = 781 \times 10^{\square}$ $(^{-}2)$
   b. $61320 = 613.20 \times 10^{\square}$ (2)
   c. $1\% = 1 \times 10^{\square}$ $(^{-}2)$
   d. $7.60 \times 10^{3} = 7.600 \times 10^{\square}$ (3)

4. Fill in the blank with the numeral of the correct number:
   a. 7m. = ____________ cm. (700)
   b. 400 cm. = ____________ mm. (4000)
   c. 100 m. = ____________ km. (0.1)
   d. 2.54 cm. = ____________ mm. (25.4)
   e. 1.5 km. = ____________ mm. (1,500,000)

Multiple Choice:

1. Which one of the following represents a number between one million and one billion?
   - a. $2(10^3 + 10^3)$
   - b. 2×10^6
   - c. 200,000
   - d. 2×10^9
   - e. 2,000,000,000 (b)

2. The product of 10^{-5} and 10^{-3} is equal to
   - a. 10^{-15}
   - b. 10^{15}
   - c. 10^8
   - d. 10^{-2}
   - e. none of these (e)

3. Which is the largest number?
   - a. 0.01
   - b. 1.4×10^{-2}
   - c. 15×10^{-4}
   - d. 15.5×10^{-4}
   - e. 0.11×10^{-2} (b)

4. 3.14×10^2 is how many times as large as 3.14?
   - a. one
   - b. two
   - c. ten
   - d. one hundred
   - e. none of these (d)

5. $10^{4} \times 10^{-4}$ is the same as

a. 10^{8}

b. 10^{-16}

c. 10^{0}

d. 10

e. none of these (c)

6. Which number is not in scientific notation?

a. 3.1×10^{0}

b. 3×10^{-5}

c. 10^{6}

d. 31×10^{2}

e. 3.1×10 (d)

7. 10^{-6} is the same as all of the following except

a. $\frac{1}{10^{6}}$

b. $10^{-4} + 10^{-2}$

c. $10^{-4} \times 10^{-2}$

d. $\frac{1}{10^{3}} \times \frac{1}{10^{3}}$

e. $10^{-(3 + 3)}$ (b)

8. 0.427 divided by 0.07 is equal to

a. 61

b. 6.1

c. 0.61

d. 0.610

e. none of these (b)

9. $\frac{1}{10^2}$ is equal to all of the following except

a. 10^{-2}

b. 0.01

c. $\frac{1}{100}$

d. $10^{-1} \times 10^{-1}$

e. 100% (e)

10. $10^6 \div 10^{-2}$ equals

a. 10^4

b. 10^{-12}

c. 10^8

d. 10^{-3}

e. none of these (c)

11. 10^0 is the same as:

a. $\frac{0}{10}$

b. 10×10

c. $10 - 10$

d. 1

e. 0 (d)

12. Which of the following is equal to 500,000?

a. 5×10^4

b. 5×10^5

c. 50×10^5

d. 10^5

e. 50^5 (b)

13. When a number between 0 and 1 is written in scientific notation, the power of 10 which is used is:

a. 0

b. 1

c. always negative.

d. always positive.

e. may be negative or positive, depending on the number. (c)

14. Which of the following is equal to $10^m \times 10^n$?

a. $10^{m + n}$

b. 10^{mn}

c. $10^{m - n}$

d. $100^{m + n}$

e. 100^{mn} (a)

The preliminary edition of *The Introduction to Secondary School Mathematics* was prepared at the writing session held at Stanford University during the summer of 1960 and was based upon Volume I of *Mathematics for Junior High School*. Parts 1 and 2 of *The Introduction to Secondary School Mathematics* were revised at Yale University in the summer of 1961 in accordance with reports of classroom effectiveness. Part 4, based on selected chapters of Volume II of *Mathematics for Junior High School* was prepared at the same time. The present revision, taking into account the classroom experience with the earlier editions during the academic years of 1961-62, was prepared at Stanford University during the summer of 1962.

The following is a list of all those who have participated in the preparation of *Introduction to Secondary School Mathematics*, Volumes I and II.

Donald R. Clarkson, North Haven Junior High School, North Haven, Connecticut

Sally Herriot, Cubberley Senior High School, Palo Alto, California

Mary T. Huggins, Jordan Junior High School, Palo Alto, California

Helen L. Hughes, Theo. Roosevelt Junior High School, Eugene, Oregon

Florence Jacobson, Albertus Magnus College, New Haven, Connecticut

Mildred Keiffer, Cincinnati Public Schools, Cincinnati, Ohio

Emma M. Lewis, Washington, D. C., Public Schools, Washington, D. C.

Muriel Mills, Hill Junior High School, Denver, Colorado

Max Peters, Wingate High School, New York City

Oscar Schaaf, South Eugene High School, Eugene, Oregon

George Schaefer, Alexis DuPont High School, Wilmington, Delaware

Veryl Schult, Washington, D. C., Public Schools, Washington, D. C.

Max A. Sobel, Montclair State College, Montclair, New Jersey

Warren Stenberg, University of Minnesota, Minneapolis, Minnesota

William C. Walsh, Proviso Township High School, Maywood, Illinois